

hep-ph/9912358
LMU-99-16

Mass and Flavor Mixing Schemes of Quarks and Leptons

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Abstract

We give an overview of recent progress in the study of fermion mass and flavor mixing phenomena. The hints exhibited by the quark and lepton mass spectra towards possible underlying flavor symmetries, from which realistic models of mass generation could be built, are emphasized. A variety of schemes of quark mass matrices at low and superhigh energy scales are described, and their consequences on flavor mixing and CP violation are discussed. Instructive patterns of lepton mass matrices, which can naturally lead to large flavor mixing angles, are explored to interpret current data on atmospheric and solar neutrino oscillations. We expect that B -meson factories and long-baseline neutrino experiments will soon shed more light on the dynamics of fermion masses, flavor mixing and CP violation.

Invited Article to appear in Vol. 45 of Prog. Part. Nucl. Phys.
Elsevier Science, The Netherlands (2000)

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1 Introduction

Since its foundation in the 1960's the standard electroweak model, which unifies the weak and electromagnetic interactions, has passed all experimental tests. Neither significant evidence for the departures from the standard model nor convincing hints for the presence of new physics has been found thus far at HERA, LEP, SLC, Tevatron and other high-energy facilities [1]. In spite of the impressive success of the standard model, many physicists believe that it does not represent the final theory, but serves merely as an effective theory originating from a more fundamental, yet unknown theoretical framework. For instance there is little understanding, within the standard model, about the intrinsic physics of the electroweak symmetry breaking, the hierarchy of charged fermion mass spectra, the vanishing or smallness of neutrino masses, and the origin of flavor mixing and CP violation. Any attempt towards gaining an insight into such problems inevitably requires significant steps to go beyond the standard model.

The investigations of fermion mass generation and flavor mixing problems, which constitute an important part of today's particle physics, can be traced back to the early 1970's, soon after the establishment of the standard electroweak model. Since then many approaches have been developed, in the contexts of different theoretical and phenomenological models. Regardless of the energy scales at which those models are built, the mechanisms for fermion mass generation and flavor mixing can roughly be classified into four different types: (a) Radiative mechanisms [2], (b) Texture zeros [3, 4], (c) Family symmetries [5, 6], and (d) Seesaw mechanisms [7]. These mechanisms cannot be regarded as disjoint from one another; rather they are related. The mechanism (d) is especially related to a natural interpretation of the smallness of neutrino masses. Phenomenologically some striking progress has been made, in particular with the help of the mechanisms (b) and (c), in specifying the quantitative relationship between flavor mixing angles and quark mass ratios [8]. From the theoretical point of view, however, our understanding of the fermion mass spectrum remains quite unsatisfactory. In a model with effective higher-dimension fermion mass operators and non-abelian family symmetries, for instance, the *observed* quark mass hierarchy is interpreted by the *assumed* texture of $U(1)$ flavor charges or Higgs-field vacuum expectation values. Another example is the determination of left-handed Majorana neutrino masses from the *assumed* textures of Dirac and (or) right-handed Majorana neutrino mass matrices through the seesaw mechanism. In many cases, the problem seems to be transferred from one place to another, leaving the model itself with few testable predictions. Before a significant breakthrough takes place on the theoretical side, the phenomenological approaches will remain to play a crucial role in interpreting new experimental data on quark mixing, CP violation, and neutrino oscillations. They are expected to provide useful hints towards discovering the full dynamics of fermion mass generation and CP violation.

This article aims at giving an overview of recent progress in the phenomenological study of fermion masses, flavor mixing and CP violation, in particular with respect to the searches for underlying discrete or continuous flavor symmetries which can lead to a realistic texture of fermion mass matrices. We are motivated not only by the theoretical significance of these topics, as briefly outlined above, but also by their experimental prospects at present and in the near future. The new e^+e^- and hadronic B -meson fac-

tories are about to open a new era to determine the flavor mixing parameters and the CP -violating phases in the quark sector to an unprecedented degree of accuracy. A variety of neutrino experiments, being done or to be done, are expected to pin down the true mechanism of neutrino oscillations and to provide a wealth of precise information about neutrino masses, lepton flavor mixing angles, and even leptonic CP violation. Such experimental developments, together with those in searching for new particles at much higher energy scales, will allow us to gain a deeper understanding of the standard model, especially the sector of Yukawa interactions.

The remaining parts of this article are organized as follows. An overview of the fermion mass spectra is given in section 2. We list the values of quark masses at the weak-interaction scale $\mu = M_Z$ and highlight the hierarchy features of both the charged lepton mass spectrum and the quark mass spectrum. For neutrinos the upper mass bounds from the direct-mass-search experiments and the mass-squared differences from the neutrino oscillation experiments are summarized. The distinctions between Dirac and Majorana neutrino masses are briefly described.

In section 3 we illustrate the main features of fermion flavor mixing and CP violation. The numbers of flavor mixing and CP -violating parameters are counted, for both quarks and leptons. The necessary and sufficient conditions of CP violation in the standard electroweak model are clarified. A geometric description of CP violation, in terms of the unitarity triangles in the complex plane, is introduced for both quark and lepton sectors. After classifying a variety of different parametrizations of the 3×3 flavor mixing matrix, we highlight a unique one which is particularly favored for the study of quark mass matrices and B -meson physics.

Section 4 is devoted to the realistic schemes of quark mass matrices. First of all we derive the flavor mixing and CP -violating parameters from a generic Hermitian texture of 3×3 quark mass matrices. Two useful symmetry limits of quark masses are taken into account, and the concept of the light-quark triangle is introduced. We then discuss an interesting Hermitian pattern with four texture zeros, and explore its consequences on flavor mixing and CP violation in detail. In particular we show that the light-quark triangle is congruent with the rescaled unitarity triangle to a good degree of accuracy. The Hermitian schemes of quark mass matrices with five texture zeros are briefly summarized. The idea of flavor democracy is stressed, and the possible breaking patterns of this symmetry are discussed in order to generate the light quark masses and the flavor mixing angles. We also look at the non-Hermitian textures of quark mass matrices by taking three typical examples, i.e., the nearest-neighbor mixing pattern, the triangular pattern, and the pure phase pattern. Finally the running effects of quark masses and flavor mixing parameters from superhigh energy scales to the weak scale are illustrated in the framework of the minimal supersymmetric standard model. A specific ansatz of quark and charged lepton mass matrices is proposed at the scale of supersymmetric grand unified theories, and its low-energy consequences on flavor mixing and CP violation is analytically calculated.

Section 5 is devoted to lepton mass matrices and neutrino oscillations. In view of current experimental data on atmospheric and solar neutrino oscillations, we make a generic classification of possible textures of 3×3 neutrino mass matrices. To be specific we study a simple model of lepton flavor mixing and CP violation based on the breaking of the charged lepton flavor democracy and the neutrino mass degeneracy. The numerous con-

sequences of this model, including nearly bi-maximal, bi-maximal and small-versus-large mixing patterns, are explored in detail for neutrino oscillations. The texture zeros of lepton mass matrices, similar to those of quark mass matrices, are taken into account to interpret the neutrino mass hierarchy and large lepton flavor mixing. In particular we emphasize that a seesaw-invariant texture of lepton mass matrices could naturally originate from a grand unified theory. We present an illustrative scheme of four-neutrino mixing, which can well accommodate the present solar, atmospheric and accelerator neutrino oscillation data. The scale dependence of the neutrino mass matrix is qualitatively described by using the renormalization-group equations. Finally we point out the essential features of leptonic CP violation in both three- and four-neutrino mixing scenarios, and discuss the possibility to measure CP and T asymmetries in the long-baseline neutrino experiments.

The conclusion and outlook are given in section 6.

2 Overview of fermion mass spectra

2.1 Charged lepton and quark masses

In the standard $SU(3) \times SU(2) \times U(1)$ model of strong, weak and electromagnetic interactions, it is the Higgs mechanism that provides a theoretically consistent framework to generate masses for gauge bosons and fermions – the latter acquire masses, after spontaneous breaking of the $SU(2)$ gauge symmetry, through the Yukawa couplings and the vacuum expectation value of the neutral Higgs field. This framework, however, can neither predict the values of fermion masses nor interpret the observed hierarchy of their spectra. Hence the three charged lepton masses and six quark masses are free parameters of the standard model. The vanishing of three neutrino masses follows as a straightforward consequence of the symmetry structure of the standard model.

The physical mass of a charged lepton is just the pole of its propagator and can directly be measured. We have [1]

$$\begin{aligned} m_e &= 0.51099907 \pm 0.00000015 \text{ MeV} , \\ m_\mu &= 105.658389 \pm 0.000034 \text{ MeV} , \\ m_\tau &= 1777.05^{+0.29}_{-0.26} \text{ MeV} . \end{aligned} \tag{2.1}$$

The fact that the mass spectrum of charged leptons is almost entirely dominated by the tau-lepton mass strongly suggests the existence of a “rank-one” limit of the corresponding mass matrix:

$$M_{0l} = C_l \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{2.2}$$

in which $m_e = m_\mu = 0$ and $m_\tau = C_l$. The realistic mass matrix M_l can be obtained if proper perturbative corrections to M_{0l} , from which the light charged leptons e and μ become massive and non-degenerate, are taken into account [9].

Since quarks are confined inside hadrons, their masses cannot directly be measured. The only way to determine the quark masses is through the study of their impact on

hadron properties. The quark mass parameters in the QCD and electroweak Lagrangians depend both on the renormalization scheme adopted to define the theory and on the scale parameter μ – this dependence reflects the fact that a bare quark is surrounded by a cloud of gluons and quark-antiquark pairs. In the limit where all quark masses vanish, the QCD Lagrangian has a $SU(3)_L \times SU(3)_R$ chiral symmetry, under which left- and right-handed quarks transform independently. The scale of dynamical chiral symmetry breaking, $\Lambda_\chi \approx 1$ GeV, can be used to distinguish between light quarks ($m < \Lambda_\chi$) and heavy quarks ($m > \Lambda_\chi$) [10]. To determine the quark mass values one may make use of the QCD perturbation theory at high energy scales, i.e., $\mu \gg \Lambda_\chi$, where nonperturbative effects such as chiral symmetry breaking are negligible.

Useful information on the mass ratios of light quarks can be obtained from analyzing properties of the light pseudoscalar mesons with the help of the chiral perturbation theory. For example, it has been argued that m_u/m_d and m_s/m_d fulfil the following relation [11]:

$$\frac{m_s/m_d}{\sqrt{1 - (m_u/m_d)^2}} = 22.7 \pm 0.08 . \quad (2.3)$$

The absolute values of m_u , m_d and m_s , usually normalized to the scale $\mu = 1$ GeV, can be extracted from QCD sum rules. The lattice gauge theory is expected to be a powerful and accurate tool for computing meson masses directly from the QCD Lagrangian, thus it provides another way to determine the light quark masses. Conservatively we list the ranges of light quark masses, allowed by current data [1, 12] and rescaled to $\mu = 1$ GeV in the modified minimal subtraction (\overline{MS}) scheme, as follows:

$$\begin{aligned} m_u(1 \text{ GeV}) &= 2 - 6.8 \text{ MeV} , \\ m_d(1 \text{ GeV}) &= 4 - 12 \text{ MeV} , \\ m_s(1 \text{ GeV}) &= 81 - 230 \text{ MeV} . \end{aligned} \quad (2.4)$$

Note that the light quark masses under discussion are the *current* masses, which have nothing to do with the *constituent* masses defined in nonrelativistic quark models.

A study of the spectrum and decays of hadrons containing heavy quarks allows one to extract useful information about the heavy quark masses. The calculations may be done with the help of the heavy quark effective theory, the QCD sum rules, the lattice gauge theory, etc.. Within the QCD perturbation theory one may define the position of the pole in the quark propagator as the quark mass m_{pol} , the so-called *pole* quark mass, which is independent of the adopted renormalization scheme. One may also define the \overline{MS} running quark mass $m(\mu)$ by regulating the QCD theory using dimensional regularization and subtracting the divergences using the \overline{MS} scheme. The relation between the pole quark mass and the running quark mass at the one-loop level of perturbative QCD corrections reads [13]

$$m_{\text{pol}} = m(m_{\text{pol}}) \left[1 + \frac{4}{3} \cdot \frac{\alpha_s(m_{\text{pol}})}{\pi} \right] , \quad (2.5)$$

where $\alpha_s(\mu)$ is the strong-interaction coupling constant. The generously allowed ranges of the charm and bottom running masses in the \overline{MS} scheme are given by [1]

$$\begin{aligned} m_c(m_c) &= 1.1 - 1.4 \text{ GeV} , \\ m_b(m_b) &= 4.1 - 4.4 \text{ GeV} ; \end{aligned} \quad (2.6)$$

and the experimentally measured value of the top mass is [1]

$$m_t = 173.8 \pm 5.2 \text{ GeV} . \quad (2.7)$$

Given the techniques used to extract the top mass at CDF and DØ [14], the mass value in (2.7) should be interpreted as the top *pole* mass.

The quark mass values given in (2.4), (2.6) and (2.7) indicate the existence of a strong mass hierarchy in both (u, c, t) and (d, s, b) quark sectors. To get the relative magnitudes of different quark masses in a physically meaningful way, one has to describe all quark masses in the same scheme and at the same scale. It is instructive to consider the light and heavy quark masses at the scale $\mu = M_Z$, the mass of the Z boson, by adopting the $\overline{\text{MS}}$ scheme. The advantage of choosing M_Z as the reference scale is two-fold: above M_Z extensions of the standard model may naturally appear; below M_Z the strong-interaction coupling constant α_s is sizable and special attention has to be paid to the running and the matching in passing a heavy quark threshold. For illustration the ranges of six quark masses given above are listed at the scale M_Z as follows:

$$\begin{aligned} m_u(M_Z) &= 0.9 - 2.9 \text{ MeV} , \\ m_c(M_Z) &= 0.53 - 0.68 \text{ GeV} , \\ m_t(M_Z) &= 168 - 180 \text{ GeV} ; \end{aligned} \quad (2.8)$$

and

$$\begin{aligned} m_d(M_Z) &= 1.8 - 5.3 \text{ MeV} , \\ m_s(M_Z) &= 35 - 100 \text{ MeV} , \\ m_b(M_Z) &= 2.8 - 3.0 \text{ GeV} . \end{aligned} \quad (2.9)$$

The hierarchical pattern of the up-type quark masses (m_u, m_c, m_t) and the down-type quark masses (m_d, m_s, m_b) is remarkable. For each quark sector the mass spectrum is dominated by the mass of the third-family quark. This property is similar to that of the charged lepton mass spectrum. It implies that the quark mass matrix M_u or M_d , like the charged lepton mass matrix M_l , is close to an interesting “rank-one” limit, in which the masses of the first two quark families vanish ($C_u = m_t$ and $C_d = m_b$):

$$M_{0q} = C_q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \quad (2.10)$$

Such a symmetry limit may be very suggestive for studying the realistic textures of quark mass matrices in the flavor basis [9]. It is also worth mentioning two approximate quantitative relations of quark masses:

$$\begin{aligned} \frac{m_u}{m_c} &\sim \frac{m_c}{m_t} \sim \lambda^4 , \\ \frac{m_d}{m_s} &\sim \frac{m_s}{m_b} \sim \lambda^2 , \end{aligned} \quad (2.11)$$

where $\lambda \approx 0.22$. The larger hierarchy of the up-type quark masses implies that they may have less significant contributions to the quark flavor mixing angles. This feature, which will be discussed in some detail in section 4, is expected to be true for most realistic models of quark mass matrices.

2.2 Massive Dirac and Majorana neutrinos

In the standard model neutrinos are assumed to be the exactly massless Weyl particles. This assumption agrees with all direct-mass-search experiments, which have set the upper bounds on masses of the primary mass eigenstates (ν_1, ν_2, ν_3) of electron, muon and tau neutrinos [1]¹:

$$\begin{aligned} m_{\nu_1} &< 15 \text{ eV} , \\ m_{\nu_2} &< 0.17 \text{ MeV} , \\ m_{\nu_3} &< 18.2 \text{ MeV} . \end{aligned} \quad (2.12)$$

However, the masslessness of neutrinos is not assured by any basic symmetry principle of particle physics. Indeed most extensions of the standard model (such as the grand unified theories) allow the existence of massive neutrinos, although the masses of three active neutrinos may be extremely smaller than those of their corresponding charged leptons.

If neutrinos have masses, they may be either Dirac or Majorana particles. A massive Dirac neutrino field consists of four independent states – left-handed and right-handed particle states (ν_L and ν_R) as well as left-handed and right-handed antiparticle states ($\bar{\nu}_L$ and $\bar{\nu}_R$). Among them ν_L and $\bar{\nu}_R$ already exist in the standard model and can take part in weak interactions. The ν_R and $\bar{\nu}_L$ states need to be introduced into the standard model as necessary ingredients to give the Dirac neutrino a mass, but they should be “sterile” in the sense that they would not take part in the normal weak interactions. A Dirac mass term, which conserves the total lepton number but violates the law of individual lepton flavor conservation, can be written as

$$-\mathcal{L}_{\text{Dirac}} = \bar{\psi}_L M_\nu^D \psi_R + \bar{\psi}_R M_\nu^{D\dagger} \psi_L , \quad (2.13)$$

where $\psi \equiv \psi_L + \psi_R$ denotes a column vector in family space of the neutrino interaction eigenstates (ν_e, ν_μ and ν_τ), and M_ν^D is the corresponding 3×3 Dirac mass matrix.

On the other hand, the neutrino ν might be a Majorana particle, which has only two independent particle states of the same mass (ν_L and $\bar{\nu}_R$, or ν_R and $\bar{\nu}_L$). By definition, a Majorana neutrino is its own antiparticle: $\nu^c \equiv C\bar{\nu}^T = e^{i\Theta}\nu$, where C denotes the charge-conjugation operator and Θ is an arbitrary real phase. A Majorana mass term, which violates both the law of total lepton number conservation and that of individual lepton flavor conservation, can be written either as

$$-\mathcal{L}_{\text{Majorana}} = \frac{1}{2} \left[\bar{\psi}_L M_\nu^M (\psi^c)_R + \overline{(\psi^c)_R} M_\nu^{M\dagger} \psi_L \right] , \quad (2.14)$$

or as

$$-\tilde{\mathcal{L}}_{\text{Majorana}} = \frac{1}{2} \left[\overline{(\psi^c)_L} \tilde{M}_\nu^M \psi_R + \bar{\psi}_R \tilde{M}_\nu^{M\dagger} (\psi^c)_L \right] , \quad (2.15)$$

where M_ν^M and \tilde{M}_ν^M stand for the symmetric 3×3 mass matrices of active and sterile Majorana neutrinos, respectively. The most general neutrino mass Lagrangian is the sum of $\mathcal{L}_{\text{Dirac}}$, $\mathcal{L}_{\text{Majorana}}$ and $\tilde{\mathcal{L}}_{\text{Majorana}}$, in which M_ν^D , M_ν^M and \tilde{M}_ν^M are $n \times n$ complex matrices.

¹The limits are kinematically obtained from the tritium β -decay ${}^3\text{H} \rightarrow {}^3\text{He} + e^- + \bar{\nu}_e$, the $\pi^+ \rightarrow \mu^+ + \nu_\mu$ decay and the $\tau \rightarrow 5\pi + \nu_\tau$ (or $\tau \rightarrow 3\pi + \nu_\tau$) decay, respectively.

Although Dirac and Majorana neutrinos have different properties, it remains extremely difficult to distinguish between them in practical high-energy experiments, if there are not right-handed currents [15]. This is easy to understand [16]: since the electroweak interactions conserve helicity (i.e., a L- or R-state remains as the L- or R-state), they are not sensitive to distinguish between Dirac and Majorana neutrino states. Presumably the only possibility to identify the type of neutrinos would be to measure the neutrinoless double beta decay, which occurs through the exchange of a Majorana neutrino between two decaying neutrons inside a nucleus and thus violates the lepton number by two units (e.g., ${}^A_Z X \rightarrow {}^A_{Z+2} X + 2e^-$). The latest upper bound on the effective neutrino mass of the $(\beta\beta)_{0\nu}$ process is [17]

$$\langle m_\nu \rangle = \sum_i \left(m_i V_{ei}^2 \right) \leq 0.2 \text{ eV} \quad (2.16)$$

at the 90% confidence level, where m_i denotes the Majorana neutrino mass of the i -th family, and V_{ei} is the element of the lepton flavor mixing matrix V . In the framework of three active neutrinos, V links the neutrino mass eigenstates $|\nu_i\rangle$ (for $i = 1, 2, 3$) to the neutrino flavor eigenstates $|\nu_\alpha\rangle$ (for $\alpha = e, \mu, \tau$).

The recent observation of the atmospheric and solar neutrino anomalies, particularly that in the Super-Kamiokande experiment [18], has provided indirect but strong evidence that neutrinos are massive and lepton flavors are mixed. Analyses of the atmospheric neutrino deficit in the framework of two-flavor neutrino oscillations yield the mass-squared difference

$$\Delta m_{\text{atm}}^2 \sim 10^{-3} \text{ eV}^2 \quad (2.17)$$

with the mixing factor $\sin^2 2\theta_{\text{atm}} > 0.8$. In contrast, there exist two different oscillation mechanisms yielding three possible solutions to the solar neutrino problem: the long wavelength vacuum oscillation (“Just-so” mechanism) with

$$\Delta m_{\text{sun}}^2 \sim 10^{-10} \text{ eV}^2 \quad (2.18)$$

and $\sin^2 2\theta_{\text{sun}} \approx 1$; and the matter-enhanced oscillation (Mikheyev-Smirnov-Wolfenstein or MSW mechanism [19]) with

$$\Delta m_{\text{sun}}^2 \sim 10^{-6} \text{ eV}^2 \quad (2.19)$$

and $\sin^2 2\theta_{\text{sun}} \sim 10^{-3} - 10^{-2}$ (small-angle solution) or with

$$\Delta m_{\text{sun}}^2 \sim 10^{-5} \text{ eV}^2 \quad (2.20)$$

and $\sin^2 2\theta_{\text{sun}} \sim 0.65 - 1$ (large-angle solution) [20]. In the framework of three-flavor neutrino oscillations, the big hierarchy between Δm_{atm}^2 and Δm_{sun}^2 together with the no observation of $\bar{\nu}_e \rightarrow \bar{\nu}_e$ oscillation in the CHOOZ experiment [21] implies that the ν_3 -component in ν_e is rather small (even negligible) and the atmospheric neutrino oscillation decouples approximately from the solar neutrino oscillation. If this simplified picture is essentially true, then the solar and atmospheric neutrino deficits are dominated respectively by the $\nu_\mu \rightarrow \nu_e$ and $\nu_\mu \rightarrow \nu_\tau$ transitions with the mass-squared differences ²

$$\begin{aligned} \Delta m_{\text{sun}}^2 &= \Delta m_{21}^2 \equiv m_2^2 - m_1^2, \\ \Delta m_{\text{atm}}^2 &= \Delta m_{32}^2 \equiv m_3^2 - m_2^2. \end{aligned} \quad (2.21)$$

²Without loss of any generality we have assumed $m_1 < m_2 < m_3$.

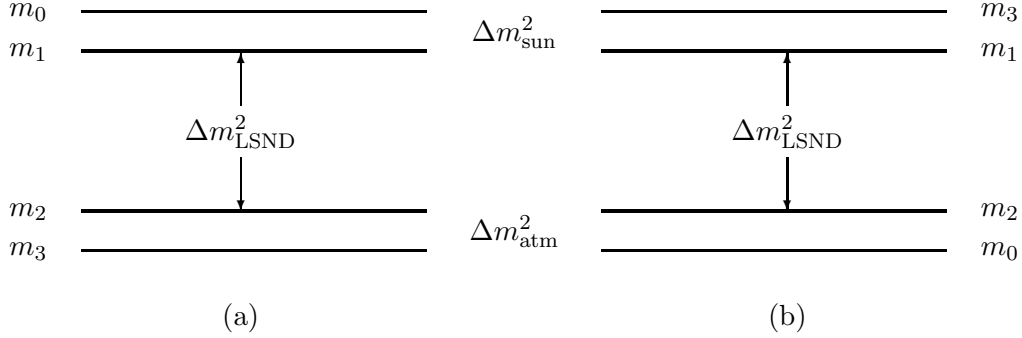


Figure 2.1: Possible neutrino mass spectra to accommodate current data on solar, atmospheric and LSND neutrino oscillations.

Nevertheless, the hierarchy of Δm_{21}^2 and Δm_{32}^2 ($\approx \Delta m_{31}^2$) can shed little light on the absolute values or the relative magnitudes of three neutrino masses. For example, either the strongly hierarchical neutrino mass spectrum ($m_1 \ll m_2 \ll m_3$) or the nearly degenerate one ($m_1 \approx m_2 \approx m_3$) is practically allowed to reproduce the “observed” gap between Δm_{21}^2 and Δm_{32}^2 .

The LSND evidence [22] for the conversion of neutrino flavors is so far the only indication of neutrino oscillations which is a *signal* instead of a *deficit*. Such evidence has been seen for both $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ and $\nu_\mu \rightarrow \nu_e$ oscillations with the mass-squared difference

$$\Delta m_{\text{LSND}}^2 \sim 1 \text{ eV}^2 \quad (2.22)$$

and the mixing factor $\sin^2 2\theta_{\text{LSND}} \sim 10^{-3} - 10^{-2}$. However, the LSND observation was not confirmed by the recent KARMEN experiment [23], which is sensitive to most of the LSND parameter space. It has to be seen how the disagreement between these two measurements is resolved. Before a further check of the LSND result which will be available in the coming years, the conservative approach is to set it aside tentatively and to concentrate on solar and atmospheric neutrino oscillations. The latter can therefore be interpreted by use of two mass-squared differences defined in (2.21).

Indeed it is extremely difficult, if not impossible [24], to accommodate the solar, atmospheric and LSND data simultaneously within the scheme of three-flavor neutrino oscillations. A more natural approach is to assume the existence of a light sterile neutrino (ν_s with mass m_0), which may produce an additional mass-squared difference with one of the three active neutrinos to fit Δm_{LSND}^2 . Recent analyses of the four-neutrino mixing patterns suggest that there exist two possible options [25]: the solar, atmospheric and LSND neutrino oscillations may be attributed respectively to (a) $\nu_e \rightarrow \nu_s$, $\nu_\mu \rightarrow \nu_\tau$ and $\nu_\mu \rightarrow \nu_e$ transitions; or to (b) $\nu_e \rightarrow \nu_\tau$, $\nu_\mu \rightarrow \nu_s$ and $\nu_\mu \rightarrow \nu_e$ transitions (see Fig. 2.1 for illustration). Note that pattern (a) seems to be more favored by the Super-Kamiokande data, but pattern (b) has not been ruled out. Either pattern involves two neutrinos in the eV mass range, which might be a suitable candidate for the hot dark matter of the universe.

The indication that the mass density of the universe is smaller than its critical value (i.e., $\Omega_m < 1$) comes in particular from direct observation of the existence of galaxies

at very high redshift, the high baryon content of galaxies, and the high-redshift Type 1A supernovae. A model which fits current data on the cosmic microwave background anisotropies and the large-scale distribution of galaxies and clusters, on the other hand, requires $\Omega_m = 1$ with about 70% cold dark matter, 20% hot dark matter and 10% baryonic matter [26]. Neutrinos with a sum of their masses in the range of a few eV could constitute the hot dark matter, which is relativistic or at a temperature lower than 1 keV. If the experimental results of solar, atmospheric and LSND neutrino oscillations would be correct, then we should be left with the hot dark matter composed of two types of massive neutrinos, as mentioned above.

If the LSND result were invalid, it is possible that the hot dark matter consists of three massive neutrinos with a mass spectrum

$$m_1 \approx m_2 \approx m_3 \sim 2 \text{ eV} . \quad (2.23)$$

The near degeneracy of three neutrino masses is of particular interest, as it could arise from the breaking of a specific flavor symmetry in which the neutrinos have an exact mass degeneracy [27, 28]:

$$M_\nu^{(0)} = m_i \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{pmatrix} \quad (2.24)$$

with $m_1 = m_2 = m_3$ and $|\eta_i| = 1$ for $i = 1, 2, 3$. For Majorana neutrinos η_i denote their CP parities. Thus $\eta_1 = \eta_2 = \eta_3$ means that $M_\nu^{(0)}$ has an exact $S(3)$ symmetry. The implications of $M_\nu^{(0)}$ and its symmetry breaking on lepton flavor mixing and neutrino oscillations will be explored in some detail in section 5. It should be noted, however, that it is impossible to accommodate current atmospheric and solar neutrino oscillation data and to satisfy the requirement of hot dark matter simultaneously, if three neutrinos have a hierarchical mass spectrum [29].

We shall not mention some other astrophysical and cosmological hints that neutrinos are probably massive [30]. Instead let us ask the question why the masses of neutrinos, if not vanishing, are so small in comparison with those of the charged leptons and quarks. This feature could be linked to the fact that neutrinos are the only known *neutral* fermions. A theoretically natural interpretation of the smallness of neutrino masses, e.g., in grand unified theories and most of other extensions of the standard model, is to assume that the active neutrinos are left-handed Majorana particles accompanied by very heavy right-handed (sterile) Majorana partners. The latter may serve to reduce the masses of the former through the well-known seesaw mechanism [7]. In the spirit of the seesaw mechanism, the left-handed Majorana neutrino mass matrix M_ν^M is given as

$$M_\nu^M = (M_\nu^D)^T (\tilde{M}_\nu^M)^{-1} (M_\nu^D) , \quad (2.25)$$

where M_ν^D and \tilde{M}_ν^M are the Dirac neutrino mass matrix and the right-handed Majorana neutrino mass matrix, respectively. If the mass eigenvalues of M_ν^D are of the order of charged lepton masses and those of \tilde{M}_ν^M are at the level of a superhigh energy scale, then the active Majorana neutrinos will acquire small (even tiny) masses. In some unified theories of quarks and leptons M_ν^D is usually taken to be the same as the up-type quark mass matrix M_u , and the mass eigenvalues of \tilde{M}_ν^M are more or less of order 10^{13} GeV. For a review of various versions of the seesaw mechanism, we refer the reader to Ref. [31].

3 Flavor mixing and CP violation

3.1 Number of fermion mixing parameters

The quark mass matrices in the Lagrangian of Yukawa interactions, M_u and M_d , can be diagonalized by the bi-unitary transformations:

$$\begin{aligned}\mathcal{M}_u &= U_{uL}^\dagger M_u U_{uR} = \text{Diag}\{m_u, m_c, m_t\}, \\ \mathcal{M}_d &= U_{dL}^\dagger M_d U_{dR} = \text{Diag}\{m_d, m_s, m_b\}.\end{aligned}\quad (3.1)$$

Such transformations, equivalent to changing quark fields from the basis of flavor eigenstates to that of mass eigenstates, introduce non-diagonal couplings into the Lagrangian of charged weak interactions, in which only the left-handed quarks take part. This 3×3 coupling matrix (the so-called Cabibbo-Kobayashi-Maskawa or CKM matrix [32, 33]), given as

$$V = U_{uL}^\dagger U_{dL}, \quad (3.2)$$

describes the mixing of quark flavors. The unitarity is a constraint, imposed by the symmetry structure of the standard model, on the flavor mixing matrix V .

Although we have taken the number of quark families to be three, a more general discussion can be made for n families of quarks. In this case the flavor mixing matrix V will be a $n \times n$ unitary matrix consisting of n^2 real parameters: $n(n-1)/2$ of them may be taken as rotation angles and the remaining are phase angles. Since the phases of quark fields are arbitrary, one can redefine them so as to rearrange the phase parameters of V . This freedom allows $2n-1$ phase angles to be absorbed. Therefore V can be described in terms of only $n^2 - (2n-1) = (n-1)^2$ parameters, among which $n(n-1)/2$ are the rotation angles and $(n-1)(n-2)/2$ are the phase angles. For the case $n=3$, we then arrive at three mixing angles and one nontrivial phase, which is responsible for CP violation.

The charged lepton and Dirac neutrino mass matrices in the flavor basis, M_l and M_ν^D , can be diagonalized in a similar way by the bi-unitary transformations:

$$\begin{aligned}\mathcal{M}_l &= U_{lL}^\dagger M_l U_{lR} = \text{Diag}\{m_e, m_\mu, m_\tau\}, \\ \mathcal{M}_\nu^D &= U_{\nu L}^\dagger M_\nu^D U_{\nu R} = \text{Diag}\{m_1, m_2, m_3\}.\end{aligned}\quad (3.3)$$

In the basis of mass eigenstates, one arrives at a 3×3 flavor mixing matrix V_l^D in the Lagrangian of charged weak interactions, in which only the left-handed leptons take part:

$$V_l^D = U_{lL}^\dagger U_{\nu L}. \quad (3.4)$$

Note that V_l^D links the neutrino mass eigenstates (ν_1, ν_2, ν_3) to the neutrino flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$ in the basis that the charged lepton mass matrix is diagonal. Similar to the quark mixing case, the 3×3 lepton mixing matrix V_l^D can be generalized to the $n \times n$ one for n families of leptons. The latter can then be parametrized in terms of $n(n-1)/2$ rotation angles and $(n-1)(n-2)/2$ phase angles.

If neutrinos are Majorana particles, however, the situation is different. In this case the neutrino mass matrix M_ν^M has the property $(M_\nu^M)^T = M_\nu^M$, i.e., M_ν^M is in general a

complex symmetric $n \times n$ matrix. The diagonalization of M_ν^M needs only a single unitary matrix \hat{U}_ν as follows ($n = 3$, for example):

$$\mathcal{M}_\nu^M = \hat{U}_\nu^T M_\nu^M \hat{U}_\nu = \text{Diag}\{m_1, m_2, m_3\}. \quad (3.5)$$

Accordingly the lepton flavor mixing matrix is given by

$$V_l^M = U_{lL}^\dagger \hat{U}_\nu, \quad (3.6)$$

where U_{lL}^\dagger has been given in (3.3) to diagonalize the charged lepton mass matrix M_l . The $n \times n$ flavor mixing matrix V_l^M consists totally of n^2 real parameters, and $n(n-1)/2$ of them can always be chosen as rotation angles. Unlike the quark or Dirac neutrino mixing case, there is no freedom to redefine phases of the Majorana neutrino fields, as Majorana particles are their own antiparticles. Hence some phases of V_l^M can be absorbed only by redefining the charged lepton fields. The number of physical phase angles left in V_l^M is $n(n+1)/2 - n = n(n-1)/2$. Thus V_l^M can be parametrized in terms of $n(n-1)/2$ rotation angles and the same number of phase angles. For the case $n = 3$, we obtain the lepton mixing matrix with 3 rotation angles and 3 CP -violating phases.

It should be noted, however, that the $n \times n$ Majorana-type flavor mixing matrix V_l^M can always be represented in the following form [34]:

$$V_l^M = V_l^D P_\nu, \quad (3.7)$$

in which V_ν^D is the $n \times n$ Dirac-type flavor mixing matrix and P_ν is a $n \times n$ diagonal phase matrix with $n-1$ nontrivial phase parameters³. The CP -violating phases in P_ν , in addition to those in V_l^D , characterize the Majorana nature of V_l^M . It is straightforward to show

$$(V_l^M)_{i\alpha} (V_l^M)_{j\beta} (V_l^M)_{i\beta}^* (V_l^M)_{j\alpha}^* = (V_l^D)_{i\alpha} (V_l^D)_{j\beta} (V_l^D)_{i\beta}^* (V_l^D)_{j\alpha}^* \quad (3.8)$$

for arbitrary indices i, j and α, β . This result implies that V_l^M and V_l^D have the same physical effects in neutrino oscillation probabilities, which depend only upon the combinations in (3.8). We then arrive at the conclusion that it is impossible to determine the type of massive neutrinos by studying the phenomena of neutrino oscillations. Only the experiments which probe transitions between left-handed and right-handed neutrino states, like the neutrinoless double beta decay, could tell whether massive neutrinos are of the Majorana type or not.

In subsequent discussions about the lepton flavor mixing and CP violation, we shall make use of the notations V_l^M and V_l^D only when it is necessary to distinguish between the Dirac and Majorana neutrinos. Otherwise V_l will simply be adopted to represent either V_l^D or V_l^M . If it is unnecessary to distinguish the lepton mixing from the quark mixing, we shall just use V to denote the flavor mixing matrix in general.

3.2 Conditions for CP violation

The quark sector of the standard electroweak model with three fermion families consists of ten free parameters: six quark masses (associated directly with M_u and M_d) and four flavor

³The elements of P_ν can be written as $(P_\nu)_{ij} = \delta_{ij} e^{i\sigma_j}$ (for $i, j = 1, \dots, n$) with $\sigma_1 = 0$.

mixing parameters (related directly to V). Since V is obtained by the diagonalization of M_u and M_d , the flavor mixing parameters are expected to depend to a great extent on the quark masses. These explicit relationships should be obvious in an underlying theory of fermion mass generation, which would be more fundamental than the standard model. Within the standard model, the conditions for CP violation can be expressed in terms of either the parameters of M_u and M_d or those of V . It is worth remarking that a double-counting of the CP -violating conditions, in terms of the parameters of both $M_{u,d}$ and V , must be avoided.

We first discuss the necessary and sufficient conditions for CP violation at the level of quark mass matrices. Although M_u and M_d are arbitrary 3×3 matrices, the products $M_u M_u^\dagger \equiv H_u$ and $M_d M_d^\dagger \equiv H_d$ are Hermitian, and each of them can be diagonalized by a single unitary transformation:

$$\begin{aligned} U_{uL}^\dagger H_u U_{uL} &= \mathcal{M}_u^2, \\ U_{dL}^\dagger H_d U_{dL} &= \mathcal{M}_d^2, \end{aligned} \quad (3.9)$$

where U_{uL} and U_{dL} have been given in (3.1). It is obvious that CP symmetry will be violated, if and only if there is at least one nontrivial phase difference between H_u and H_d . In other words, $\text{Im}(H_{uij} H_{dij}^*) \neq 0$ (for $i, j = 1, 2, 3$ and $i < j$) is the necessary and sufficient condition for CP violation in the standard model. If one defines a commutator for H_u and H_d , $[H_u, H_d] \equiv i \mathcal{C}$, then it is easy to show

$$\frac{\mathcal{C}_{ii}}{2} = \text{Im}(H_{uij} H_{dij}^*) + \text{Im}(H_{uik} H_{dik}^*) \quad (3.10)$$

for $i, j, k = 1, 2, 3$ but $i \neq j \neq k$. Clearly $\mathcal{C}_{ii} \neq 0$ if CP symmetry is violated [8]. Note that CP symmetry would be conserved, if two quarks with the same charge were degenerate in mass. The reason is simply that the freedom induced by the mass degeneracy of two up- or down-type quarks allows one to rearrange the matrix elements of H_u or H_d and remove all possible phase differences between H_u and H_d (or between M_u and M_d).

Now we discuss the condition for CP violation at the level of the flavor mixing matrix. Of course CP symmetry is violated, if V contains a nontrivial complex phase which cannot be removed from the redefinition of quark-field phases. The most appropriate measure of CP violation, due to the unitarity of V , is the rephasing-invariant parameter \mathcal{J} [35] defined through

$$\text{Im}(V_{i\alpha} V_{j\beta} V_{i\beta}^* V_{j\alpha}^*) = \mathcal{J} \sum_{k, \gamma} (\varepsilon_{ijk} \varepsilon_{\alpha\beta\gamma}) , \quad (3.11)$$

in which each Latin subscript (i, j or k) runs over the up-type quarks (u, c, t) and each Greek subscript (α, β or γ) runs over the down-type quarks (d, s, b). In terms of the matrix elements of V , \mathcal{J}^2 can be given as

$$\begin{aligned} \mathcal{J}^2 &= |V_{i\alpha}|^2 |V_{j\beta}|^2 |V_{i\beta}|^2 |V_{j\alpha}|^2 - \frac{1}{4} \left(1 + |V_{i\alpha}|^2 |V_{j\beta}|^2 \right. \\ &\quad \left. + |V_{i\beta}|^2 |V_{j\alpha}|^2 - |V_{i\alpha}|^2 - |V_{j\beta}|^2 - |V_{i\beta}|^2 - |V_{j\alpha}|^2 \right) . \end{aligned} \quad (3.12)$$

This result implies that the information about CP violation can in principle be extracted from the moduli of the flavor mixing matrix elements. Note again that any mass degeneracy between two quarks with the same charge results in CP conservation. If two up- or down-type quarks were degenerate in mass, then any linear combination of their mass eigenstates via an arbitrary unitary matrix would remain a mass eigenstate. This additional freedom allows us to redefine the phases of relevant quark fields and eliminate the single nontrivial (CP -violating) phase in V . The resultant flavor mixing matrix is real (with either three or two nonvanishing mixing angles), then $\mathcal{J} = 0$ holds, i.e., CP symmetry is conserved. In fact one can show, as done in Ref. [8], that the magnitude of \mathcal{J} is proportional to the product of the mass-squared differences $(m_i^2 - m_j^2)$ and $(m_\alpha^2 - m_\beta^2)$, where the subscripts (i, j) and (α, β) run over (u, c, t) and (d, s, b) respectively, for arbitrary mass matrices M_u and M_d . Therefore any mass degeneracy between two quarks with the same charge would lead to $\mathcal{J} = 0$.

In short, the conditions for CP violation in the quark sector can be set at either the level of quark mass matrices or that of the flavor mixing matrix [8]. A double-counting, arising from a combination of the conditions obtained from two different levels, must be avoided. It should in particular be noted that the determinant of \mathcal{C} and the rephasing-invariant parameter \mathcal{J} , which are related to each other through [35]

$$\text{Det } \mathcal{C} = -2\mathcal{J} \prod_{i < j} (m_i^2 - m_j^2) \prod_{\alpha < \beta} (m_\alpha^2 - m_\beta^2), \quad (3.13)$$

contain the same information about CP violation. In reality the quark masses of both up and down sectors have been found to exhibit a clear hierarchy, and all elements of the flavor mixing matrix are nonvanishing. Therefore the realistic condition for CP violation is only associated with the existence of one nontrivial phase parameter in V , which in turn requires (at least) one nontrivial phase difference between M_u and M_d (or equivalently between H_u and H_d).

The conditions for CP violation in weak interactions of charged leptons and Dirac neutrinos can be discussed in the same way as outlined above. Similar to \mathcal{J} , a rephasing-invariant measure of leptonic CP asymmetry, \mathcal{J}_l^D , can be defined in terms of elements of the flavor mixing matrix V_l^D . Therefore the necessary and sufficient condition for leptonic CP violation is just $\mathcal{J}_l^D \neq 0$.

If neutrinos are of the Majorana type, however, one has to take care of two additional CP -violating phases from P_ν in (3.7). In this case the necessary and sufficient condition for CP violation is that there exist nontrivial phase differences between the mass matrices M_l and M_ν^M , i.e., the flavor mixing matrix V_l^M contains at least one nontrivial complex phase. In analogy to the definition of \mathcal{J}_l^D , a CP -violating parameter \mathcal{J}_l^M which is independent of the phase rearrangement of charged lepton fields, can be defined. It should be noted that $\mathcal{J}_l^M = 0$ is only a necessary condition for CP invariance in weak interactions of the charged leptons and Majorana neutrinos. Taking (3.8) into account, we arrive at

$$\mathcal{J}_l^M = \mathcal{J}_l^D \equiv \mathcal{J}_l. \quad (3.14)$$

Hence only \mathcal{J}_l appears in the transition probabilities of neutrino oscillations.

Finally let us emphasize that the mass degeneracy of two Majorana neutrinos may not lead to the vanishing of CP violation in a model with the family number $n \geq 3$, provided

they have the opposite CP parities. It has been shown in Ref. [36] that the 3×3 flavor mixing matrix V_l^M consists of two rotation angles and one CP -violating phase, if all three Majorana neutrinos are degenerate in mass but one of them has a different CP parity from the other two. However, the mass degeneracy of any two Dirac neutrinos would make the CP -violating phase in V_l^D removable and thus lead to CP conservation. It should be noted that there would be no neutrino oscillations, if neutrino masses were degenerate.

3.3 Unitarity triangles of quark mixing

A geometric description of the quark flavor mixing phenomenon turns out to be instructive. The unitarity of the 3×3 flavor mixing matrix V can be expressed by two sets of orthogonality relations and two sets of normalization conditions for all nine matrix elements:

$$\begin{aligned} \sum_{\alpha} (V_{i\alpha}^* V_{j\alpha}) &= \delta_{ij} , \\ \sum_i (V_{i\alpha}^* V_{i\beta}) &= \delta_{\alpha\beta} , \end{aligned} \quad (3.15)$$

where the Latin indices run over the up-type quarks (u, c, t) and the Greek indices over the down-type quarks (d, s, b). In the complex plane the six orthogonality relations in (3.15) define six triangles, the so-called unitarity triangles (see Fig. 3.1 for illustration). In general these six triangles have eighteen different sides and nine different inner (or outer) angles. Unitarity requires that all six triangles have the same area amounting to $\mathcal{J}/2$ [35], where \mathcal{J} is just the rephasing-invariant measure of CP violation defined in (3.11). If CP were an exact symmetry, $\mathcal{J} = 0$ would hold, and those unitarity triangles would collapse into lines in the complex plane.

In addition to \mathcal{J} , there exist other two characteristic quantities of V , resulting from its normalization conditions in (3.15). They are the off-diagonal asymmetries \mathcal{A}_1 and \mathcal{A}_2 , respectively about the axes $V_{ud}\text{-}V_{cs}\text{-}V_{tb}$ and $V_{ub}\text{-}V_{cs}\text{-}V_{td}$ [37]:

$$\begin{aligned} \mathcal{A}_1 &\equiv |V_{us}|^2 - |V_{cd}|^2 = |V_{cb}|^2 - |V_{ts}|^2 = |V_{td}|^2 - |V_{ub}|^2 , \\ \mathcal{A}_2 &\equiv |V_{us}|^2 - |V_{cb}|^2 = |V_{cd}|^2 - |V_{ts}|^2 = |V_{tb}|^2 - |V_{ud}|^2 . \end{aligned} \quad (3.16)$$

Clearly $\mathcal{A}_1 = 0$ (or $\mathcal{A}_2 = 0$) would imply that the flavor mixing matrix were symmetric about the $V_{ud}\text{-}V_{cs}\text{-}V_{tb}$ (or $V_{ub}\text{-}V_{cs}\text{-}V_{td}$) axis. Geometrically this corresponds to the congruence between two unitarity triangles, i.e.,

$$\begin{aligned} \mathcal{A}_1 = 0 &\implies \triangle_u \cong \triangle_d , \quad \triangle_c \cong \triangle_s , \quad \triangle_t \cong \triangle_b ; \\ \mathcal{A}_2 = 0 &\implies \triangle_u \cong \triangle_b , \quad \triangle_c \cong \triangle_s , \quad \triangle_t \cong \triangle_d . \end{aligned} \quad (3.17)$$

In reality, however, both \mathcal{A}_1 and \mathcal{A}_2 are nonvanishing. In terms of the Wolfenstein parameters of V [38], the three measurables \mathcal{J} , \mathcal{A}_1 and \mathcal{A}_2 can be expressed as follows:

$$\begin{aligned} \mathcal{J} &\approx A^2 \lambda^6 \eta , \\ \mathcal{A}_1 &\approx A^2 \lambda^6 (1 - 2\rho) , \\ \mathcal{A}_2 &\approx \lambda^2 (1 - A^2 \lambda^2) . \end{aligned} \quad (3.18)$$

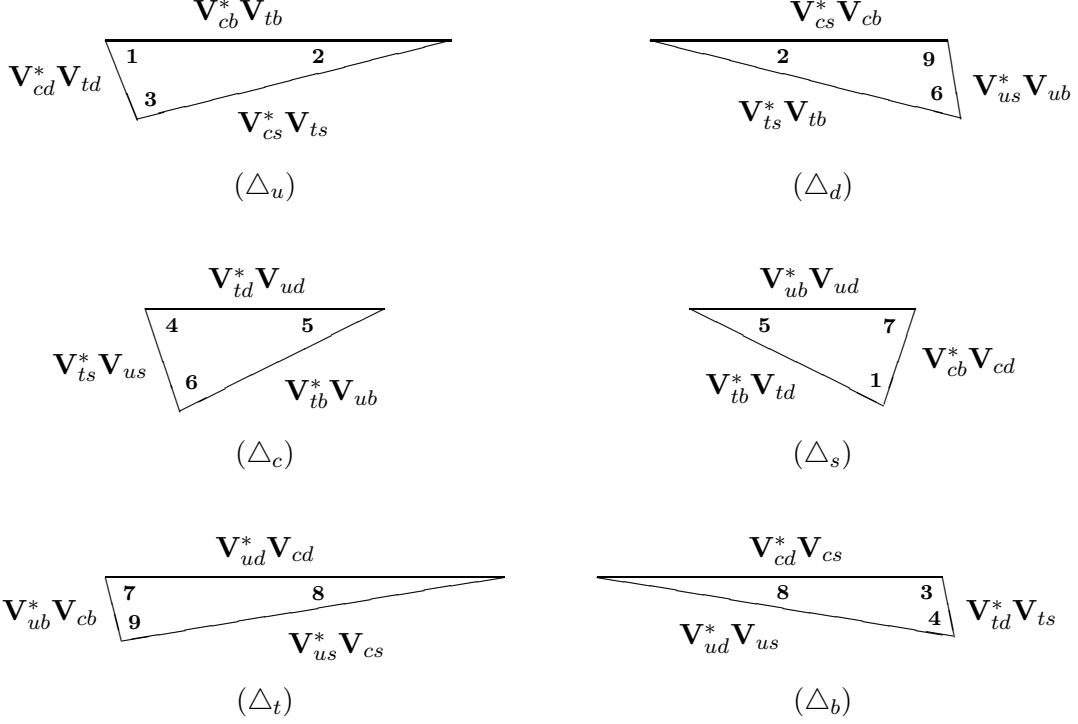


Figure 3.1: Unitarity triangles of the flavor mixing matrix in the complex plane. Each triangle is named in terms of the quark flavor that does not manifest in its three sides.

It is clear that $\mathcal{A}_2 \gg \mathcal{A}_1 \sim \mathcal{J} \sim 10^{-5}$. Therefore V deviates only a little bit from an exactly symmetric matrix (about the V_{ud} - V_{cs} - V_{tb} axis). With $\mathcal{A}_1 > 0$ one arrives at a clear hierarchy among nine elements of V :

$$\begin{aligned}
|V_{tb}| > |V_{ud}| > |V_{cs}| &\gg |V_{us}| > |V_{cd}| \\
&\gg |V_{cb}| > |V_{ts}| \\
&\gg |V_{td}| > |V_{ub}| > 0.
\end{aligned} \tag{3.19}$$

Indeed $|V_{ub}| \neq 0$ is a necessary condition for CP violation in the standard model.

It is worth remarking that the triangle Δ_s is, among six unitarity triangles, of particular interest for the study of B -meson physics and CP violation. Indeed three inner angles of Δ_s , usually denoted as

$$\begin{aligned}
\alpha &= \arg \left(-\frac{V_{tb}^* V_{td}}{V_{ub}^* V_{ud}} \right), \\
\beta &= \arg \left(-\frac{V_{cb}^* V_{cd}}{V_{tb}^* V_{td}} \right), \\
\gamma &= \arg \left(-\frac{V_{ub}^* V_{ud}}{V_{cb}^* V_{cd}} \right),
\end{aligned} \tag{3.20}$$

can directly be measured from CP asymmetries in specific B decay modes. On the other hand, these rephasing-invariant observables can be predicted from specific models of quark mass matrices. Therefore the unitarity triangle Δ_s may serve as a phenomenological link between the theoretical understanding and the experimental measurement of quark flavor mixing and CP violation.

3.4 Unitarity triangles of lepton mixing

A similar geometric description as that discussed above for quark mixing can be introduced for the 3×3 lepton flavor mixing matrix V_l . The six leptonic unitarity triangles, which have eighteen different sides, nine different inner angles and the identical area (amounting to $\mathcal{J}_l/2$), are illustrated in Fig. 3.2. Our present knowledge on lepton flavor mixing is rather poor, hence it remains difficult to quantify the magnitude of leptonic CP violation as well as the off-diagonal asymmetries of V_l . Useful information about the sides and angles of the leptonic unitarity triangles are likely to be obtained, in the near future, from the long-baseline neutrino experiments. In particular one might determine the three inner angles of the triangle Δ_3 , i.e.,

$$\begin{aligned}\alpha_l &= \arg \left(-\frac{V_{e1}^* V_{e2}}{V_{\mu 1}^* V_{\mu 2}} \right) , \\ \beta_l &= \arg \left(-\frac{V_{\mu 1}^* V_{\mu 2}}{V_{\tau 1}^* V_{\tau 2}} \right) , \\ \gamma_l &= \arg \left(-\frac{V_{\tau 1}^* V_{\tau 2}}{V_{e1}^* V_{e2}} \right) ;\end{aligned}\tag{3.21}$$

and then test the self-consistency of the lepton flavor mixing and of CP violation (e.g., $\alpha_l + \beta_l + \gamma_l \neq \pi$ would imply the presence of new physics which violates the unitarity of the 3×3 lepton flavor mixing matrix).

To be more specific, we write down the transition probabilities of three different neutrino flavors in the vacuum:

$$P(\nu_\alpha \rightarrow \nu_\beta) = -4 \sum_{i < j} \left[\text{Re} \left(V_{\alpha i} V_{\beta j} V_{\alpha j}^* V_{\beta i}^* \right) \sin^2 F_{ij} \right] + 8 \mathcal{J}_l \prod_{i < j} (\sin F_{ij}) \tag{3.22}$$

with $(\alpha, \beta) = (e, \mu), (\mu, \tau)$ or (τ, e) ; $i, j = 1, 2, 3$; and $F_{ij} = 1.27 \Delta m_{ij}^2 L/E$, where L is the distance between the neutrino source and the detector (in unit of km) and E denotes the neutrino beam energy (in unit of GeV). Note that the CP -violating term ($\propto \mathcal{J}_l$) in $P(\nu_\alpha \rightarrow \nu_\beta)$ is independent of the neutrino flavor indices α and β . As the negative neutrino mass-squared differences Δm_{12}^2 , Δm_{13}^2 and Δm_{23}^2 can be related to the positive solar and atmospheric neutrino mass-squared differences given in (2.21), we simplify (3.22) to

$$\begin{aligned}P(\nu_e \rightarrow \nu_\mu) &= 4|V_{e3}|^2|V_{\mu 3}|^2 \sin^2 F_{\text{atm}} - 4\text{Re} \left(V_{e1} V_{\mu 2} V_{e2}^* V_{\mu 1}^* \right) \sin^2 F_{\text{sun}} \\ &\quad - 8 \mathcal{J}_l \sin F_{\text{sun}} \sin^2 F_{\text{atm}} , \\ P(\nu_\mu \rightarrow \nu_\tau) &= 4|V_{\mu 3}|^2|V_{\tau 3}|^2 \sin^2 F_{\text{atm}} - 4\text{Re} \left(V_{\mu 1} V_{\tau 2} V_{\mu 2}^* V_{\tau 1}^* \right) \sin^2 F_{\text{sun}}\end{aligned}$$

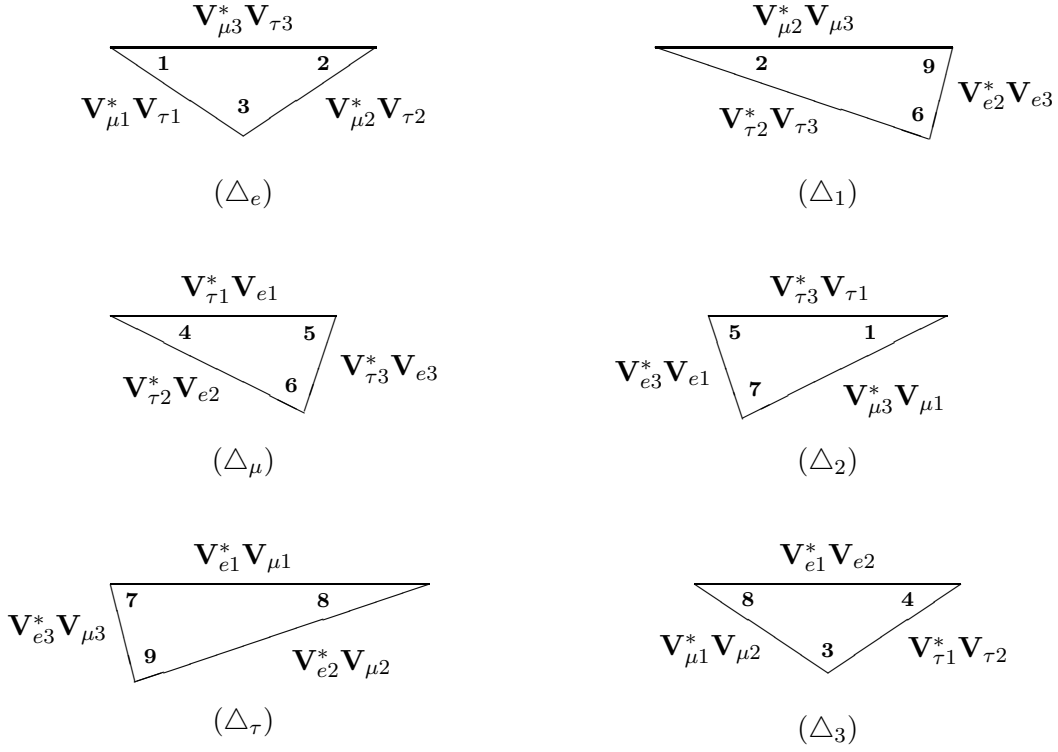


Figure 3.2: Unitarity triangles of the lepton flavor mixing matrix in the complex plane. Each triangle is named by the flavor index that does not manifest in its three sides.

$$\begin{aligned}
P(\nu_\tau \rightarrow \nu_e) = & 4|V_{\tau 3}|^2 |V_{e 3}|^2 \sin^2 F_{\text{atm}} - 8\mathcal{J}_l \sin F_{\text{sun}} \sin^2 F_{\text{atm}} , \\
& - 4\text{Re}(V_{\tau 1} V_{e 2} V_{\tau 2}^* V_{e 1}^*) \sin^2 F_{\text{sun}} \\
& - 8\mathcal{J}_l \sin F_{\text{sun}} \sin^2 F_{\text{atm}} , \quad (3.23)
\end{aligned}$$

where $F_{\text{atm}} = F_{32} \approx F_{31}$ and $F_{\text{sun}} = F_{21}$ measure the oscillation frequencies of atmospheric and solar neutrinos, respectively. If the length of the baseline satisfies the condition $L \sim E/\Delta m_{\text{sun}}^2$ (i.e., $F_{\text{sun}} \sim 1$), which is only possible for the large-angle MSW solution to the solar neutrino problem, the CP -conserving quantities $\text{Re}(V_{\alpha 1} V_{\beta 2} V_{\alpha 2}^* V_{\beta 1}^*)$ and the CP -violating parameter \mathcal{J}_l could (in principle) be determined from (3.23) for changing values of the neutrino beam energy E . In this case we arrive at

$$\begin{aligned}
\tan \alpha_l &= -\frac{\mathcal{J}_l}{\text{Re}(V_{e 1} V_{\mu 2} V_{e 2}^* V_{\mu 1}^*)} , \\
\tan \beta_l &= -\frac{\mathcal{J}_l}{\text{Re}(V_{\mu 1} V_{\tau 2} V_{\mu 2}^* V_{\tau 1}^*)} , \\
\tan \gamma_l &= -\frac{\mathcal{J}_l}{\text{Re}(V_{\tau 1} V_{e 2} V_{\tau 2}^* V_{e 1}^*)} . \quad (3.24)
\end{aligned}$$

As the CP -violating effect is universal in the three different neutrino transitions, one needs only to measure \mathcal{J}_l from the probability asymmetry between $\nu_\mu \rightarrow \nu_e$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ (or

$\nu_e \rightarrow \nu_\mu$) [39]. In practice, however, the earth-induced matter effects, which are possible to fake the genuine CP -violating signals in the long-baseline neutrino experiments, have to be taken into account [40]. A more detailed discussion about leptonic CP violation will be given in section 5.6.

Finally it is worth mentioning that $|V_{e3}|$, $|V_{\mu3}|$ and $|V_{\tau3}|$ can well be determined in the first-round long-baseline neutrino experiments, in which the scale of the neutrino mass-squared differences is set to Δm_{atm}^2 (i.e., the terms proportional to $\sin^2 F_{\text{sun}}$ and \mathcal{J}_l in (3.23) are safely negligible). This will allow a check of the unitarity of V_l , i.e., $|V_{e3}|^2 + |V_{\mu3}|^2 + |V_{\tau3}|^2 = 1$. If $|V_{e1}|$ and $|V_{e2}|$ are measured in the solar neutrino experiment, one will be able to test the unitarity condition $|V_{e1}|^2 + |V_{e2}|^2 + |V_{e3}|^2 = 1$. At this stage all three mixing angles of V_l can be determined. To isolate the CP -violating phase and make a full test of the unitarity, however, more delicate long-baseline neutrino experiments are needed.

3.5 Classification of different parametrizations

The 3×3 flavor mixing matrix V can be expressed in terms of four independent parameters, which are usually taken as three rotation angles and one CP -violating phase angle. A number of parametrizations, different from the original Kobayashi-Maskawa form [33], have been proposed in the literature [38, 41]. To adopt a specific parametrization is of course somehow arbitrary, since from a mathematical point of view the different parametrizations are equivalent. Nevertheless it is quite likely that the underlying physics responsible for flavor mixing and CP violation becomes more transparent in a particular parametrization than in the others [42]. For this reason, we find it useful to give a classification of all possible parametrizations [43] (in terms of three angles and one phase) and point out the most appropriate one for the phenomenology of mass matrices, flavor mixing and CP violation.

If the flavor mixing matrix V is first assumed to be a real orthogonal matrix, it can in general be written as a product of three matrices R_{12} , R_{23} and R_{31} , which describe simple rotations in the (1,2), (2,3) and (3,1) planes:

$$\begin{array}{ccc} R_{12}(\theta) & R_{23}(\sigma) & R_{31}(\tau) \\ \hline \begin{pmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\sigma & s_\sigma \\ 0 & -s_\sigma & c_\sigma \end{pmatrix} & \begin{pmatrix} c_\tau & 0 & s_\tau \\ 0 & 1 & 0 \\ -s_\tau & 0 & c_\tau \end{pmatrix} \end{array} \quad (3.25)$$

where $s_\theta \equiv \sin \theta$, $c_\theta \equiv \cos \theta$, and so on. Clearly these rotation matrices do not commute with one another. There exist twelve different ways to arrange products of these matrices such that the most general orthogonal matrix R can be obtained. Note that the matrix $R_{ij}^{-1}(\omega)$ plays an equivalent role as $R_{ij}(\omega)$ in constructing R , because of $R_{ij}^{-1}(\omega) = R_{ij}(-\omega)$. Note also that $R_{ij}(\omega)R_{ij}(\omega') = R_{ij}(\omega + \omega')$ holds, thus $R_{ij}(\omega)R_{ij}(\omega')R_{kl}(\omega'')$ or $R_{kl}(\omega'')R_{ij}(\omega)R_{ij}(\omega')$ cannot cover the whole space of a 3×3 orthogonal matrix and should be excluded. Explicitly six of the twelve different forms of R belong to the type

$$R = R_{ij}(\theta) R_{kl}(\sigma) R_{ij}(\theta'), \quad (3.26)$$

in which the rotation in the (i, j) plane occurs twice; and the other six belong to the type

$$R = R_{ij}(\theta) R_{kl}(\sigma) R_{mn}(\tau) , \quad (3.27)$$

where rotations take place in three different planes. Although all the twelve combinations represent the most general orthogonal matrices, only nine of them are structurally different. The reason is that the products $R_{ij}R_{kl}R_{ij}$ and $R_{ij}R_{mn}R_{ij}$ (with $ij \neq kl \neq mn$) in (3.26) are correlated with each other, leading essentially to the same form for R [43]. Thus only three of R in (3.26) need be treated as independent choices. We then draw the conclusion that there exist *nine* different forms for the orthogonal matrix R (see Table 3.1).

We proceed to introduce the CP -violating phase, denoted by φ . The complex rotation matrices should be unitary, such that a unitary flavor mixing matrix results. There are several different ways for φ to enter R_{12} , R_{23} or R_{31} . Then the flavor mixing matrix V can be constructed, as a product of three rotation matrices, by use of one complex R_{ij} and two real ones. Note that the location of the CP -violating phase in V can be arranged by redefining the quark field phases, thus it does not play an essential role in classifying different parametrizations. We find that it is always possible to locate the phase parameter φ in a 2×2 submatrix of V , in which each element is a sum of two terms with the relative phase φ . The remaining five elements of V are real in such a phase assignment. Accordingly we arrive at nine distinctive parametrizations of the flavor mixing matrix V as listed in Table 3.1, where the complex rotation matrices $R_{12}(\theta, \varphi)$, $R_{23}(\sigma, \varphi)$ and $R_{31}(\tau, \varphi)$ are obtained directly from the real ones in (3.25) with the replacement $1 \rightarrow e^{-i\varphi}$.

Some instructive relations of each parametrization, together with the expression for the \mathcal{J} parameter defined in (3.11), are also given in Table 3.1. One can see that the parametrizations $P2$ and $P3$ correspond to the Kobayashi-Maskawa [33] and Maiani [41] parametrizations, although different notations for the CP -violating phase and three mixing angles are adopted here. The latter is indeed equivalent to the “standard” parametrization advocated by the Particle Data Group [1]. This can be seen clearly if one makes three transformations of quark-field phases: $c \rightarrow c e^{-i\varphi}$, $t \rightarrow t e^{-i\varphi}$, and $b \rightarrow b e^{-i\varphi}$. In addition, the parametrization $P1$ is just the one proposed recently by the authors [42]. The specific advantages of this parametrization for the study of quark mass matrices and B -meson physics will become obvious in section 3.6 and section 4.

Besides the aforementioned parametrizations, which take advantage of three rotation angles and one CP -violating phase, there are some other ways to parametrize the 3×3 flavor mixing matrix. A representation based on a hierarchical expansion was proposed by Wolfenstein [38]:

$$V \approx \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} , \quad (3.28)$$

in which $\lambda \approx 0.22$, $A \approx 0.83$, and the magnitudes of ρ and η are smaller than one. Note that at the accuracy level of (3.28) the CP -violating parameter η appears only in two smallest elements (V_{ub} and V_{td}), whose magnitudes are suppressed by λ^3 . If the three sides of the unitarity triangle \triangle_s are rescaled by $V_{cd}V_{cb}^*$, then the resultant triangle has three vertices $(0, 0)$, $(1, 0)$ and (ρ, η) in the complex plane [1]. To discuss more precise

Table 3.1: Classification of different parametrizations for the flavor mixing matrix.

| Parametrization | Useful relations |
|--|--|
| $P1: \quad V = R_{12}(\theta) R_{23}(\sigma, \varphi) R_{12}^{-1}(\theta')$ $\begin{pmatrix} s_\theta s_{\theta'} c_\sigma + c_\theta c_{\theta'} e^{-i\varphi} & s_\theta c_{\theta'} c_\sigma - c_\theta s_{\theta'} e^{-i\varphi} & s_\theta s_\sigma \\ c_\theta s_{\theta'} c_\sigma - s_\theta c_{\theta'} e^{-i\varphi} & c_\theta c_{\theta'} c_\sigma + s_\theta s_{\theta'} e^{-i\varphi} & c_\theta s_\sigma \\ -s_{\theta'} s_\sigma & -c_{\theta'} s_\sigma & c_\sigma \end{pmatrix}$ | $\mathcal{J} = s_\theta c_\theta s_{\theta'} c_{\theta'} s_\sigma^2 c_\sigma \sin \varphi$ $\tan \theta = V_{ub}/V_{cb} $ $\tan \theta' = V_{td}/V_{ts} $ $\cos \sigma = V_{tb} $ |
| $P2: \quad V = R_{23}(\sigma) R_{12}(\theta, \varphi) R_{23}^{-1}(\sigma')$ $\begin{pmatrix} c_\theta & s_\theta c_{\sigma'} & -s_\theta s_{\sigma'} \\ -s_\theta c_\sigma & c_\theta c_\sigma c_{\sigma'} + s_\theta s_{\sigma'} e^{-i\varphi} & -c_\theta c_\sigma s_{\sigma'} + s_\theta c_{\sigma'} e^{-i\varphi} \\ s_\theta s_\sigma & -c_\theta s_\sigma c_{\sigma'} + c_\sigma s_{\sigma'} e^{-i\varphi} & c_\theta s_\sigma s_{\sigma'} + c_\sigma c_{\sigma'} e^{-i\varphi} \end{pmatrix}$ | $\mathcal{J} = s_\theta^2 c_\theta s_\sigma c_\sigma s_{\sigma'} c_{\sigma'} \sin \varphi$ $\cos \theta = V_{ud} $ $\tan \sigma = V_{td}/V_{cd} $ $\tan \sigma' = V_{ub}/V_{us} $ |
| $P3: \quad V = R_{23}(\sigma) R_{31}(\tau, \varphi) R_{12}(\theta)$ $\begin{pmatrix} c_\theta c_\tau & s_\theta c_\tau & s_\tau \\ -c_\theta s_\sigma s_\tau - s_\theta c_\sigma e^{-i\varphi} & -s_\theta s_\sigma s_\tau + c_\theta c_\sigma e^{-i\varphi} & s_\sigma c_\tau \\ -c_\theta c_\sigma s_\tau + s_\theta s_\sigma e^{-i\varphi} & -s_\theta c_\sigma s_\tau - c_\theta s_\sigma e^{-i\varphi} & c_\sigma c_\tau \end{pmatrix}$ | $\mathcal{J} = s_\theta c_\theta s_\sigma c_\sigma s_\tau c_\tau^2 \sin \varphi$ $\tan \theta = V_{us}/V_{ud} $ $\tan \sigma = V_{cb}/V_{tb} $ $\sin \tau = V_{ub} $ |
| $P4: \quad V = R_{12}(\theta) R_{31}(\tau, \varphi) R_{23}^{-1}(\sigma)$ $\begin{pmatrix} c_\theta c_\tau & c_\theta s_\sigma s_\tau + s_\theta c_\sigma e^{-i\varphi} & c_\theta c_\sigma s_\tau - s_\theta s_\sigma e^{-i\varphi} \\ -s_\theta c_\tau & -s_\theta s_\sigma s_\tau + c_\theta c_\sigma e^{-i\varphi} & -s_\theta c_\sigma s_\tau - c_\theta s_\sigma e^{-i\varphi} \\ -s_\tau & s_\sigma c_\tau & c_\sigma c_\tau \end{pmatrix}$ | $\mathcal{J} = s_\theta c_\theta s_\sigma c_\sigma s_\tau c_\tau^2 \sin \varphi$ $\tan \theta = V_{cd}/V_{ud} $ $\tan \sigma = V_{ts}/V_{tb} $ $\sin \tau = V_{td} $ |
| $P5: \quad V = R_{31}(\tau) R_{12}(\theta, \varphi) R_{31}^{-1}(\tau')$ $\begin{pmatrix} c_\theta c_\tau c_{\tau'} + s_\tau s_{\tau'} e^{-i\varphi} & s_\theta c_\tau & -c_\theta c_\tau s_{\tau'} + s_\tau c_{\tau'} e^{-i\varphi} \\ -s_\theta c_{\tau'} & c_\theta & s_\theta s_{\tau'} \\ -c_\theta s_\tau c_{\tau'} + c_\tau s_{\tau'} e^{-i\varphi} & -s_\theta s_\tau & c_\theta s_\tau s_{\tau'} + c_\tau c_{\tau'} e^{-i\varphi} \end{pmatrix}$ | $\mathcal{J} = s_\theta^2 c_\theta s_\tau c_\tau s_{\tau'} c_{\tau'} \sin \varphi$ $\cos \theta = V_{cs} $ $\tan \tau = V_{ts}/V_{us} $ $\tan \tau' = V_{cb}/V_{cd} $ |
| $P6: \quad V = R_{12}(\theta) R_{23}(\sigma, \varphi) R_{31}(\tau)$ $\begin{pmatrix} -s_\theta s_\sigma s_\tau + c_\theta c_\tau e^{-i\varphi} & s_\theta c_\sigma & s_\theta s_\sigma c_\tau + c_\theta s_\tau e^{-i\varphi} \\ -c_\theta s_\sigma s_\tau - s_\theta c_\tau e^{-i\varphi} & c_\theta c_\sigma & c_\theta s_\sigma c_\tau - s_\theta s_\tau e^{-i\varphi} \\ -c_\sigma s_\tau & -s_\sigma & c_\sigma c_\tau \end{pmatrix}$ | $\mathcal{J} = s_\theta c_\theta s_\sigma c_\sigma^2 s_\tau c_\tau \sin \varphi$ $\tan \theta = V_{us}/V_{cs} $ $\sin \sigma = V_{ts} $ $\tan \tau = V_{td}/V_{tb} $ |
| $P7: \quad V = R_{23}(\sigma) R_{12}(\theta, \varphi) R_{31}^{-1}(\tau)$ $\begin{pmatrix} c_\theta c_\tau & s_\theta & -c_\theta s_\tau \\ -s_\theta c_\sigma c_\tau + s_\sigma s_\tau e^{-i\varphi} & c_\theta c_\sigma & s_\theta c_\sigma s_\tau + s_\sigma c_\tau e^{-i\varphi} \\ s_\theta s_\sigma c_\tau + c_\sigma s_\tau e^{-i\varphi} & -c_\theta s_\sigma & -s_\theta s_\sigma s_\tau + c_\sigma c_\tau e^{-i\varphi} \end{pmatrix}$ | $\mathcal{J} = s_\theta c_\theta^2 s_\sigma c_\sigma s_\tau c_\tau \sin \varphi$ $\sin \theta = V_{us} $ $\tan \sigma = V_{ts}/V_{cs} $ $\tan \tau = V_{ub}/V_{ud} $ |
| $P8: \quad V = R_{31}(\tau) R_{12}(\theta, \varphi) R_{23}(\sigma)$ $\begin{pmatrix} c_\theta c_\tau & s_\theta c_\sigma c_\tau - s_\sigma s_\tau e^{-i\varphi} & s_\theta s_\sigma c_\tau + c_\sigma s_\tau e^{-i\varphi} \\ -s_\theta & c_\theta c_\sigma & c_\theta s_\sigma \\ -c_\theta s_\tau & -s_\theta c_\sigma s_\tau - s_\sigma c_\tau e^{-i\varphi} & -s_\theta s_\sigma s_\tau + c_\sigma c_\tau e^{-i\varphi} \end{pmatrix}$ | $\mathcal{J} = s_\theta c_\theta^2 s_\sigma c_\sigma s_\tau c_\tau \sin \varphi$ $\sin \theta = V_{cd} $ $\tan \sigma = V_{cb}/V_{cs} $ $\tan \tau = V_{td}/V_{ud} $ |
| $P9: \quad V = R_{31}(\tau) R_{23}(\sigma, \varphi) R_{12}^{-1}(\theta)$ $\begin{pmatrix} -s_\theta s_\sigma s_\tau + c_\theta c_\tau e^{-i\varphi} & -c_\theta s_\sigma s_\tau - s_\theta c_\tau e^{-i\varphi} & c_\sigma s_\tau \\ s_\theta c_\sigma & c_\theta c_\sigma & s_\sigma \\ -s_\theta s_\sigma c_\tau - c_\theta s_\tau e^{-i\varphi} & -c_\theta s_\sigma c_\tau + s_\theta s_\tau e^{-i\varphi} & c_\sigma c_\tau \end{pmatrix}$ | $\mathcal{J} = s_\theta c_\theta s_\sigma c_\sigma^2 s_\tau c_\tau \sin \varphi$ $\tan \theta = V_{cd}/V_{cs} $ $\sin \sigma = V_{cb} $ $\tan \tau = V_{ub}/V_{tb} $ |

experimental data on quark mixing and CP violation, modified versions of the Wolfenstein parametrization with higher accuracy have been proposed in the literature [44].

Furthermore one can parametrize V in terms of the moduli of four independent matrix elements, in terms of four independent angles of the unitarity triangles, or in terms of four characteristic quantities of V [45]. The choice of any four parameters is arbitrary, and different choices may be mathematically convenient for different physical purposes.

3.6 A unique description of flavor mixing

From a mathematical point of view, all different parametrizations of the flavor mixing matrix are equivalent. However, this is not the case if we take the hierarchical structure of the quark mass spectrum and its implications on the flavor mixing phenomenon into account. It is well known that both the observed quark mass spectrum and the observed values of the flavor mixing parameters exhibit a striking hierarchical structure. The latter can be understood in a natural way as the consequence of a specific pattern of chiral symmetries, whose breaking causes the towers of different masses to appear step by step. Such a chiral evolution of the mass matrices leads, as argued in Ref. [42], to a specific way to describe the flavor mixing (i.e., the parametrization *P1* in Table 3.1):

$$\begin{aligned} V &= \begin{pmatrix} c_u & s_u & 0 \\ -s_u & c_u & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\varphi} & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} \begin{pmatrix} c_d & -s_d & 0 \\ s_d & c_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} s_u s_d c + c_u c_d e^{-i\varphi} & s_u c_d c - c_u s_d e^{-i\varphi} & s_u s \\ c_u s_d c - s_u c_d e^{-i\varphi} & c_u c_d c + s_u s_d e^{-i\varphi} & c_u s \\ -s_d s & -c_d s & c \end{pmatrix}, \end{aligned} \quad (3.29)$$

where $s_u \equiv \sin \theta_u$, $s_d \equiv \sin \theta_d$, $c \equiv \cos \theta$, and so on. The three mixing angles may all be arranged to lie in the first quadrant, and the phase parameter φ generally takes values from 0 to 2π . This parametrization follows automatically from the generic Hermitian mass matrices M_u and M_d , if one imposes the constraints from the chiral symmetries and the quark mass hierarchy on them (see section 4.1 for detailed discussions). Therefore it is particularly useful for the study of realistic quark mass matrices. We shall see later on that this representation also makes the properties of flavor mixing and CP violation more transparent and proves to be very convenient for the phenomenology of B -meson decays.

The three mixing angles θ , θ_u and θ_d have direct physical meanings. The angle θ describes the mixing between the second and third families. We shall refer to this mixing (involving t and b quarks) as the “heavy quark mixing”. The angle θ_u primarily describes the u - c mixing and will be denoted as the “u-channel mixing”. The angle θ_d primarily describes the d - s mixing, and will be denoted as the “d-channel mixing”. Clearly there exists an asymmetry between the mixing of the first and second families and that of the second and third families, which in our view reflects interesting details of the underlying dynamics of flavor mixing. The heavy quark mixing is a combined effect, involving both charge $+2/3$ and charge $-1/3$ quarks, while the u- or d-channel mixing proceeds primarily in the charge $+2/3$ or charge $-1/3$ sector. Therefore an experimental determination of θ_u and θ_d would give us useful information about the underlying patterns of up and down mass matrices.

The angles θ , θ_u and θ_d are related in a very simple way to observable quantities of B -meson physics. For example, θ is related to the rate of the semileptonic decay $B \rightarrow D^* l \nu_l$; θ_u is associated with the ratio of the decay rate of $B \rightarrow (\pi, \rho) l \nu_l$ to that of $B \rightarrow D^* l \nu_l$; and θ_d can be determined from the ratio of the mass difference between two B_d mass eigenstates to that between two B_s mass eigenstates. From (3.29) one can find the following exact relations [42]:

$$\begin{aligned}\tan \theta_u &= \left| \frac{V_{ub}}{V_{cb}} \right|, \\ \tan \theta_d &= \left| \frac{V_{td}}{V_{ts}} \right|, \\ \sin \theta &= \sqrt{|V_{ub}|^2 + |V_{cb}|^2}.\end{aligned}\tag{3.30}$$

These results show that the parametrization (3.29) is particularly useful for B -meson physics. A global analysis of current experimental data on $|\epsilon_K|$, $|V_{ub}/V_{cb}|$, ΔM_{B_d} and ΔM_{B_s} , as already done in Ref. [46], yields $\theta = 2.30^\circ \pm 0.09^\circ$, $\theta_u = 4.87^\circ \pm 0.86^\circ$, $\theta_d = 11.71^\circ \pm 1.09^\circ$, and $\varphi = 91.1^\circ \pm 11.8^\circ$. The smallness of θ or θ_u implies the approximate decoupling between two different flavor mixing channels, while the fact $\varphi \approx \pi/2$ might have a deeper meaning for CP violation in the standard model (see section 4).

The dynamics of flavor mixing can easily be understood by considering certain limiting cases in (3.29). In the limit $\theta \rightarrow 0$ the flavor mixing exists only between the first and second families, described by a single (overall) mixing angle [32] – the Cabibbo angle θ_C (i.e., $s_C \equiv \sin \theta_C = |V_{us}| = |V_{cd}|$):

$$s_C = \left| s_u c_d - c_u s_d e^{-i\varphi} \right|.\tag{3.31}$$

Note that the Cabibbo angle (or the matrix element V_{us} or V_{cd}) is indeed a superposition of two terms including a phase. This feature comes up naturally in the hierarchical approach of quark mass matrices [3, 4], since the limit $\theta \rightarrow 0$ is essentially equivalent to the heavy quark limit $m_t \rightarrow \infty$, $m_b \rightarrow \infty$. Note also that (3.31) defines a triangle in the complex plane, denoted as the light-quark triangle (see Fig. 3.3(a) for illustration). The limit $\theta \rightarrow 0$, in which a variety of interesting patterns of quark mass matrices predict [8]

$$\begin{aligned}\tan \theta_u &= \sqrt{\frac{m_u}{m_c}}, \\ \tan \theta_d &= \sqrt{\frac{m_d}{m_s}},\end{aligned}\tag{3.32}$$

is very close to the reality (see section 4.3). It assures that the light-quark triangle is approximately congruent with the rescaled unitarity triangle Δ_s , as to be shown later.

As we have pointed out, the unitarity triangle Δ_s is of particular interest for B -meson physics and CP violation. Rescaling the three sides of Δ_s by V_{cb}^* , we obtain

$$|V_{ub}^* V_{ud}| : |V_{cb}^* V_{cd}| : |V_{tb}^* V_{td}| \approx s_u c_d : s_C : s_d\tag{3.33}$$

as a good approximation (with an error of a few percent). The rescaled unitarity triangle Δ_s is illustrated in Fig. 3.3(b). Comparing this triangle with the light-quark triangle

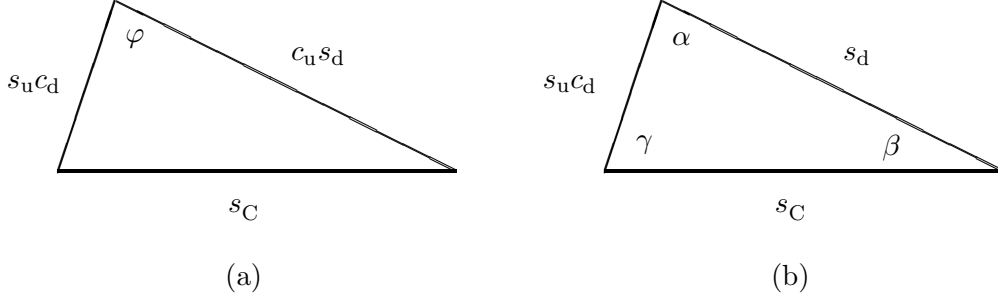


Figure 3.3: The light-quark triangle (a) and the rescaled unitarity triangle (b) in the complex plane.

shown in Fig. 3.3(a), we find that they are indeed congruent with each other to a high degree of accuracy [42, 47]⁴. The congruent relation between these two triangles is particularly interesting, since the light-quark triangle is essentially a feature of the physics of the first two quark families, while the unitarity triangle is linked to all three families and describes CP violation. As a straightforward result, $\alpha \approx \varphi$ holds. The other two angles of the unitarity triangle (i.e., β and γ) can then be predicted in terms of φ and the sides of the light-quark triangles, which may be simple functions of light quark mass ratios as indicated in (3.32). Note that both the light-quark triangle and the rescaled unitarity triangle are approximately independent of the renormalization-group effects [8, 47]. Hence information about the CP -violating angles α , β and γ obtained at low-energy scales can directly be extended to an arbitrary high-energy scale, at which a dynamical understanding of quark mass generation and CP violation might be available.

Taking the relationship in (3.33) and the approximate congruency between the unitarity triangles \triangle_c and \triangle_s (see Fig. 3.1) into account, we find that the parametrization (3.29) takes an instructive leading-order form:

$$V \approx \begin{pmatrix} e^{-i\alpha} & s_C e^{i\gamma} & s_u s \\ s_C e^{i\beta} & 1 & s \\ -s_d s & -s & 1 \end{pmatrix}. \quad (3.34)$$

The values of α , β and γ can therefore be given in terms of s_u , s_d and s_C .

Finally it is worth pointing out that there exist simple relations between the mixing parameters in (3.29) and those in (3.28). In particular $\varphi = 90^\circ$ implies $\eta^2 \approx \rho(1 - \rho)$, on the condition $0 < \rho < 1$ [42]. In this interesting case, of course, the flavor mixing matrix can fully be described in terms of only three independent parameters.

In summary, the parametrization (3.29) has many distinctions and advantages compared with other representations. We therefore highlight its usefulness in the description of flavor mixing and CP violation, in particular, for the studies of quark mass matrices and B -meson physics.

⁴Strictly, the mixing angles θ_u and θ_d in the light-quark triangle should be redefined in accordance with the limit $\theta \rightarrow 0$. We have omitted this redefinition, because corrections from $\theta \neq 0$ are tiny. This point will be seen more clearly in sections 4.2 and 4.3, where we derive both the light-quark triangle and the unitarity triangle from a realistic texture of quark mass matrices.

4 Realistic schemes of quark mass matrices

4.1 Generic Hermitian mass matrices

For a deeper understanding of the quark flavor mixing and CP -violating phenomena, it is desirable to study the properties of the quark mass matrices $M_{u,d}$, whose textures are completely unknown within the standard electroweak model. A theory more fundamental than the standard model should be able to determine M_u and M_d exclusively, such that the associated physical parameters (six quark masses, three flavor mixing angles, and one CP -violating phase) could be predicted. Attempts in this direction, e.g., those starting from supersymmetric grand unified theories and from superstring theories, have so far proved not to be very successful [48]. Phenomenologically the natural approach is to look for the simplest pattern of M_u and M_d , which can result in self-consistent and experimentally-favored relations between quark masses and flavor mixing parameters. The discrete flavor symmetries hidden in such mass matrix textures might finally provide useful hints towards the dynamics of quark mass generation and CP violation in a more fundamental theoretical framework.

Before discussing a variety of specific patterns, we first make some analyses of the general quark mass matrices and their consequences for flavor mixing and CP violation. It is worth remarking that the mass matrices M_u and M_d can be taken to be Hermitian, without loss of generality, in the standard model or its extensions which have no flavor-changing right-handed currents. The essential point is that the right-handed fields in the Lagrangians of such models are $SU(2)$ singlets. Therefore it is always possible to choose a suitable basis of the right-handed quarks, so that the resultant up- and down-type mass matrices become Hermitian. Such a redefinition of the right-handed fields does not alter the rest of the Lagrangians, i.e., it keeps the generality of the relevant physics of the model. For this reason we shall simply assume M_u and M_d to be Hermitian mass matrices in the following.

A general Hermitian mass matrix M_q (for $q = u$ or d) can be written as

$$M_q = \begin{pmatrix} E_q & D_q & F_q \\ D_q^* & C_q & B_q \\ F_q^* & B_q^* & A_q \end{pmatrix}. \quad (4.1)$$

Note that physics is invariant under a common unitary transformation of M_u and M_d , i.e., $M_q \rightarrow SM_q S^\dagger$, where S is an arbitrary unitary matrix. Using this freedom, one can always arrange F_u and F_d to vanish and obtain the following form of M_q [42]:

$$M_q = \begin{pmatrix} E_q & D_q & \mathbf{0} \\ D_q^* & C_q & B_q \\ \mathbf{0} & B_q^* & A_q \end{pmatrix}. \quad (4.2)$$

The condition $F_u = F_d = 0$, which imposes four constraints on S , can always be fulfilled, as S generally consists of nine parameters (three mixing angles and six complex phases). This implies that by going from (4.1) to (4.2) the generality of the physics remains. The basis in which the mass matrices take the form (4.2) is a particularly interesting basis of the quark flavor space. The texture zeros of M_u and M_d in this basis allows the absence

of direct mixing between the heavy quark (t or b) and the light quark (u or d). Such a feature seems quite natural from the phenomenological point of view, but it does not imply any special relations among mass eigenvalues and flavor mixing parameters. Note that M_u and M_d in (4.2) totally have twelve nontrivial parameters (ten moduli and two phase differences), thus no definite prediction can be obtained for the flavor mixing matrix.

It is worth remarking that the specific description of flavor mixing given in (3.29) can naturally be obtained from the diagonalization of $M_{u,d}$ in (4.2). The latter can in the absence of complex phases be diagonalized by a 3×3 orthogonal matrix, described only by two rotation angles in the hierarchy limit of quark masses [49]. First, the off-diagonal element B_q is rotated away by a rotation matrix R_{23} between the second and third families. Then the element D_q is rotated away by a transformation R_{12} between the first and second families. No rotation between the first and third families is necessary in either the limit $m_u \rightarrow 0$, $m_d \rightarrow 0$ or the limit $m_t \rightarrow \infty$, $m_b \rightarrow \infty$. In reality one needs an additional transformation R_{31} with a tiny rotation angle to fully diagonalize M_q . Note, however, that the rotation sequence $(R_{12}^u R_{23}^u)(R_{12}^d R_{23}^d)^T$ is enough to describe the 3×3 real flavor mixing matrix, as the effects of R_{31}^u and R_{31}^d can always be absorbed into this sequence through redefining the relevant rotation angles. By introducing a complex phase angle into the rotation combination $(R_{23}^u)(R_{23}^d)^T$, we are then able to arrive at the representation of quark flavor mixing in (3.29). Although we have derived this particular parametrization in a heuristic way from the hierarchical mass matrices in (4.2), we should like to emphasize that (3.29) is a model-independent description of any 3×3 flavor mixing matrix.

Now we start from the Hermitian mass matrices (4.2) to derive the flavor mixing angles $(\theta_u, \theta_d, \theta)$ and the CP -violating phase (φ) . Note that M_q can be decomposed into $M_q = P_q^\dagger \overline{M}_q P_q$, where

$$\overline{M}_q = \begin{pmatrix} E_q & |D_q| & \mathbf{0} \\ |D_q| & C_q & |B_q| \\ \mathbf{0} & |B_q| & A_q \end{pmatrix} \quad (4.3)$$

is a real symmetric matrix, and $P_q = \text{Diag}\{1, e^{i\phi_{D_q}}, e^{i(\phi_{B_q} + \phi_{D_q})}\}$ with $\phi_{D_q} = \arg(D_q)$ and $\phi_{B_q} = \arg(B_q)$ is a diagonal phase matrix. For simplicity we shall neglect the subscript “ q ” in the following, when it is unnecessary to distinguish between the up and down quark sectors. \overline{M} can be diagonalized by use of the orthogonal transformation $O^T \overline{M} O = \text{Diag}\{\lambda_1, \lambda_2, \lambda_3\}$, where λ_i ($i = 1, 2, 3$) are quark mass eigenvalues and may be either positive or negative. As a result, we have

$$\begin{aligned} \sum_{i=1}^3 \lambda_i &= A + C + E, \\ \prod_{i=1}^3 \lambda_i &= ACE - A|D|^2 - E|B|^2, \\ \sum_{i=1}^3 \lambda_i^2 &= A^2 + 2|B|^2 + C^2 + 2|D|^2 + E^2. \end{aligned} \quad (4.4)$$

It is a simple exercise to solve the nine matrix elements of O in terms of the parameters

of quark mass matrices. Explicitly, three diagonal elements of O read [8]⁵ :

$$\begin{aligned} O_{11} &= \left[1 + \left(\frac{\lambda_1 - E}{|D|} \right)^2 + \left(\frac{|B|}{|D|} \cdot \frac{\lambda_1 - E}{\lambda_1 - A} \right)^2 \right]^{-1/2}, \\ O_{22} &= \left[1 + \left(\frac{|D|}{\lambda_2 - E} \right)^2 + \left(\frac{|B|}{\lambda_2 - A} \right)^2 \right]^{-1/2}, \\ O_{33} &= \left[1 + \left(\frac{\lambda_3 - A}{|B|} \right)^2 + \left(\frac{|D|}{|B|} \cdot \frac{\lambda_3 - A}{\lambda_3 - E} \right)^2 \right]^{-1/2}; \end{aligned} \quad (4.5)$$

and then six off-diagonal elements of O can be obtained from the relations

$$\begin{aligned} O_{2i} &= \frac{\lambda_i - E}{|D|} O_{1i}, \\ O_{3i} &= \frac{|B|}{\lambda_i - A} O_{2i}. \end{aligned} \quad (4.6)$$

The flavor mixing matrix turns out to be $V \equiv O_u^T (P_u P_d^\dagger) O_d$. More specifically, we have

$$V_{i\alpha} = O_{1i}^u O_{1\alpha}^d + O_{2i}^u O_{2\alpha}^d e^{i\phi_1} + O_{3i}^u O_{3\alpha}^d e^{i(\phi_1 + \phi_2)}, \quad (4.7)$$

where $\phi_1 \equiv \phi_{D_u} - \phi_{D_d}$ and $\phi_2 \equiv \phi_{B_u} - \phi_{B_d}$; and the Latin subscript i and the Greek subscript α run over (u, c, t) and (d, s, b) , respectively.

To link the parameters of flavor mixing with those of quark mass matrices in an analytically concise way, we first define four dimensionless quantities:

$$\begin{aligned} X_u &\equiv \left| \frac{|D_u|}{\lambda_1^u - E_u} \cdot \frac{|D_d| (\lambda_3^d - A_d)}{|B_d| (\lambda_3^d - E_d)} + \frac{\lambda_3^d - A_d}{|B_d|} e^{i\phi_1} + \frac{|B_u|}{\lambda_1^u - A_u} e^{i(\phi_1 + \phi_2)} \right|, \\ Y_u &\equiv \left| \frac{|D_u|}{\lambda_2^u - E_u} \cdot \frac{|D_d| (\lambda_3^d - A_d)}{|B_d| (\lambda_3^d - E_d)} + \frac{\lambda_3^d - A_d}{|B_d|} e^{i\phi_1} + \frac{|B_u|}{\lambda_2^u - A_u} e^{i(\phi_1 + \phi_2)} \right|; \end{aligned} \quad (4.8)$$

and (X_d, Y_d) can directly be obtained from (X_u, Y_u) through the subscript exchange $u \leftrightarrow d$ in (4.8). After a lengthy but straightforward calculation, we arrive at

$$\begin{aligned} \tan \theta_u &= \frac{O_{21}^u}{O_{22}^u} \cdot \frac{X_u}{Y_u}, \\ \tan \theta_d &= \frac{O_{21}^d}{O_{22}^d} \cdot \frac{X_d}{Y_d}, \end{aligned} \quad (4.9)$$

and

$$\begin{aligned} \sin \theta &= \left[(O_{21}^u)^2 X_u^2 + (O_{22}^u)^2 Y_u^2 \right]^{1/2} O_{33}^d, \\ &= \left[(O_{21}^d)^2 X_d^2 + (O_{22}^d)^2 Y_d^2 \right]^{1/2} O_{33}^u, \end{aligned} \quad (4.10)$$

⁵Here and hereafter, the off-diagonal elements B and D are both taken to be nonvanishing. The relevant calculations will somehow be simplified if one of them vanishes.

where O_{21} , O_{22} and O_{33} for up and down sectors have been given in (4.5) and (4.6). Also an indirect relation between φ and $\phi_{1,2}$ can be obtained as follows:

$$\cos \varphi = \frac{s_u^2 c_d^2 c^2 + c_u^2 s_d^2 - |V_{us}|^2}{2s_u c_u s_d c_d c}, \quad (4.11)$$

where

$$|V_{us}| = O_{11}^u O_{22}^d \left| \frac{|D_d|}{\lambda_2^d - E_d} + \frac{\lambda_1^u - E_u}{|D_u|} e^{i\phi_1} \left(1 + \frac{|B_u|}{\lambda_1^u - A_u} \cdot \frac{|B_d|}{\lambda_2^d - A_d} e^{i\phi_2} \right) \right|. \quad (4.12)$$

If the hierarchies of the matrix elements and mass eigenvalues of $M_{u,d}$ are taken into account, one can see that the effect of ϕ_2 on $|V_{us}|$ is strongly suppressed and thus negligible. Fitting $|V_{us}|$ with current data should essentially determine the magnitude of ϕ_1 . Note also that the terms associated with ϕ_1 and ϕ_2 may primarily be cancelled in the ratios X_u/Y_u and X_d/Y_d due to the hierarchical structures of M_u and M_d , hence the dependence of θ_u and θ_d on $\phi_{1,2}$ could be negligible in the leading order approximation. Although the mixing angle θ may be sensitive to ϕ_1 and ϕ_2 (or one of them), its smallness indicated by current data makes the factor $\cos \theta$ in the denominator of $\cos \varphi$ completely negligible. As a result, (4.11) and (4.12) imply that the CP -violating phase φ depends dominantly on ϕ_1 through $|V_{us}|$, unless the magnitude of ϕ_1 is very small. Without fine-tuning, we find that a delicate numerical analysis does support the argument made here, i.e., ϕ_2 plays a negligible role for CP violation in V , because of the hierarchy of quark masses. The observed CP violation is linked primarily to the phases in the (1,2) and (2,1) elements of the quark mass matrices.

4.2 Symmetry limits of quark masses

Given Hermitian mass matrices of the form (4.2), one may consider two useful symmetry limits of quark masses and analyze their corresponding consequences on the quark flavor mixing angles.

The chiral limit of quark masses In the limit $m_u \rightarrow 0$, $m_d \rightarrow 0$ (“chiral limit”), where both the up- and down-type quark mass matrices have zeros in the positions (1, 1), (1, 2), (2, 1), (1, 3) and (3, 1), the flavor mixing angles θ_u and θ_d vanish. Only the θ rotation affecting the heavy quark sector remains, i.e., the flavor mixing matrix effectively takes the form

$$\hat{V} = \begin{pmatrix} \cos \hat{\theta} & \sin \hat{\theta} \\ -\sin \hat{\theta} & \cos \hat{\theta} \end{pmatrix}, \quad (4.13)$$

where $\hat{\theta}$ denotes the value of θ which one obtains in the limit $\theta_u \rightarrow 0$, $\theta_d \rightarrow 0$. We see that \hat{V} is a real orthogonal matrix, arising naturally from V in the chiral limit.

The flavor mixing angle $\hat{\theta}$ can be derived from Hermitian quark mass matrices of the following general form (in the limit $m_u \rightarrow 0$, $m_d \rightarrow 0$):

$$\hat{M}_q = \begin{pmatrix} \hat{C}_q & \hat{B}_q \\ \hat{B}_q^* & \hat{A}_q \end{pmatrix}, \quad (4.14)$$

where $|\hat{A}_q| \gg |\hat{B}_q|, |\hat{C}_q|$; and $q = u$ (up) or d (down). Note that the phase difference between \hat{B}_u and \hat{B}_d , denoted as $\kappa \equiv \arg(\hat{B}_u) - \arg(\hat{B}_d)$, has no effect on CP symmetry in the chiral limit, but it may affect the magnitude of $\hat{\theta}$. It is known that current data on the top-quark mass and the B -meson lifetime disfavor the special case $\hat{C}_u = \hat{C}_d = 0$ for \hat{M}_u and \hat{M}_d , hence we take $\hat{C}_q \neq 0$ and define a ratio $\hat{r}_q \equiv |\hat{B}_q|/\hat{C}_q$ for convenience. Then we can obtain the flavor mixing angle $\hat{\theta}$, in terms of the quark mass ratios m_c/m_t , m_s/m_b and the parameters \hat{r}_u , \hat{r}_d , by diagonalizing the mass matrices in (4.14). In the next-to-leading order approximation, $\sin \hat{\theta}$ reads

$$\sin \hat{\theta} = \left| \hat{r}_d \frac{m_s}{m_b} (1 - \hat{\delta}_d) - \hat{r}_u \frac{m_c}{m_t} (1 - \hat{\delta}_u) e^{i\kappa} \right|, \quad (4.15)$$

where two correction terms are given by

$$\begin{aligned} \hat{\delta}_u &= \left(1 + \hat{r}_u^2\right) \frac{m_c}{m_t}, \\ \hat{\delta}_d &= \left(1 + \hat{r}_d^2\right) \frac{m_s}{m_b}. \end{aligned} \quad (4.16)$$

In view of the fact $m_s/m_b \sim \mathcal{O}(10)$ m_c/m_t from current data [1], we find that the flavor mixing angle $\hat{\theta}$ is primarily linked to m_s/m_b provided $|\hat{r}_u| \approx |\hat{r}_d|$. Note that in specific models, e.g., those describing the mixing between the second and third families as an effect related to the breaking of an underlying “democratic symmetry” [50], the ratios \hat{r}_u and \hat{r}_d are purely algebraic numbers (such as $|\hat{r}_u| = |\hat{r}_d| = 1/\sqrt{2}$ or $\sqrt{2}$).

For illustration, we take $\hat{r}_u = \hat{r}_d \equiv \hat{r}$ to fit the experimental result $\sin \hat{\theta} = 0.040 \pm 0.002$ with the typical inputs $m_b/m_s = 26 - 36$ and $m_t/m_c \sim 250$. It is found that the favored value of $|\hat{r}|$ varies in the range $1.0 - 2.5$, dependent weakly on the phase parameter κ .

Note that both m_s/m_b and m_c/m_t evolve with the energy scale (see section 4.7 for a detailed discussion), therefore $\hat{\theta}$ itself is also scale-dependent.

The heavy quark limit The limit $m_t \rightarrow \infty$, $m_b \rightarrow \infty$ is subsequently referred to as the “heavy quark limit”. In this limit, in which the $(3, 3)$ elements of the up- and down-type mass matrices formally approach infinity but all other matrix elements are fixed, the angle θ vanishes. The flavor mixing matrix, which is nontrivial only in the light quark sector, takes the form:

$$\begin{aligned} \tilde{V} &= \begin{pmatrix} \tilde{c}_u & \tilde{s}_u \\ -\tilde{s}_u & \tilde{c}_u \end{pmatrix} \begin{pmatrix} e^{-i\tilde{\varphi}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{c}_d & -\tilde{s}_d \\ \tilde{s}_d & \tilde{c}_d \end{pmatrix} \\ &= \begin{pmatrix} \tilde{s}_u \tilde{s}_d + \tilde{c}_u \tilde{c}_d e^{-i\tilde{\varphi}} & \tilde{s}_u \tilde{c}_d - \tilde{c}_u \tilde{s}_d e^{-i\tilde{\varphi}} \\ \tilde{c}_u \tilde{s}_d - \tilde{s}_u \tilde{c}_d e^{-i\tilde{\varphi}} & \tilde{c}_u \tilde{c}_d + \tilde{s}_u \tilde{s}_d e^{-i\tilde{\varphi}} \end{pmatrix}. \end{aligned} \quad (4.17)$$

where $\tilde{s}_u = \sin \tilde{\theta}_u$, $\tilde{c}_u = \cos \tilde{\theta}_u$, and so on. The angles $\tilde{\theta}_u$ and $\tilde{\theta}_d$ are the values for θ_u and θ_d obtained in the heavy quark limit. Since the (t, b) system is decoupled from the (c, s) and (u, d) systems, the flavor mixing can be described as in the case of two families. Therefore the mixing matrix \tilde{V} is effectively given in terms of only a single rotation angle, the Cabbibo angle θ_C :

$$\sin \theta_C = |\tilde{s}_u \tilde{c}_d - \tilde{c}_u \tilde{s}_d e^{-i\tilde{\varphi}}|. \quad (4.18)$$

Of course $\tilde{V}(\theta_C)$ is essentially a real matrix, because its complex phases can always be rotated away by redefining the quark fields.

We stress that the heavy quark limit, which carries the flavor mixing matrix V to its simplified form \tilde{V} , is not far from the reality, since $1 - c \approx 0.1\%$ holds [42]. Therefore θ_u , θ_d and φ are expected to approach $\tilde{\theta}_u$, $\tilde{\theta}_d$ and $\tilde{\varphi}$ rapidly, as $\theta \rightarrow 0$, corresponding to $m_t \rightarrow \infty$ and $m_b \rightarrow \infty$. However, the concrete limiting behavior depends on the specific algebraic structure of the up- and down-type mass matrices. If two Hermitian mass matrices have the parallel hierarchy with texture zeros in the (1,1) (2,2), (1,3) and (3,1) elements [4], for example, the magnitude of θ is suppressed by the terms proportional to $1/\sqrt{m_t}$ and $1/\sqrt{m_b}$; and if the (2,2) elements are kept nonvanishing and comparable in magnitude with the (2,3) and (3,2) elements [8], then θ is dependent on $1/m_t$ and $1/m_b$.

The angles $\tilde{\theta}_u$ and $\tilde{\theta}_d$ as well as the phase $\tilde{\varphi}$ are well-defined quantities in the heavy quark limit. The physical meaning of these quantities can be seen more clearly, if we take the up and down mass matrices to be of the following specific and realistic structure in the symmetry limit (i.e., $m_t \rightarrow \infty$ and $m_b \rightarrow \infty$) [3]:

$$\tilde{M}_q = \begin{pmatrix} \mathbf{0} & \tilde{B}_q \\ \tilde{B}_q^* & \tilde{A}_q \end{pmatrix}. \quad (4.19)$$

The diagonalization of \tilde{M}_u and \tilde{M}_d leads to the Cabibbo-type mixing

$$\sin \theta_C = |R_u - R_d e^{-i\psi}|, \quad (4.20)$$

where

$$\begin{aligned} R_u &= \sqrt{\frac{m_u}{m_u + m_c}} \sqrt{\frac{m_s}{m_d + m_s}}, \\ R_d &= \sqrt{\frac{m_c}{m_u + m_c}} \sqrt{\frac{m_d}{m_d + m_s}}, \end{aligned} \quad (4.21)$$

and $\psi \equiv \arg(\tilde{B}_u) - \arg(\tilde{B}_d)$ denotes the relative phase between the off-diagonal elements \tilde{B}_u and \tilde{B}_d (in the limit $m_u \rightarrow 0$ this phase can be absorbed through a redefinition of the quark fields). We find that the same structure for the Cabibbo-type mixing matrix has been obtained as in the decoupling limit discussed above. If we set

$$\begin{aligned} \tan \tilde{\theta}_u &= \sqrt{\frac{m_u}{m_c}}, \\ \tan \tilde{\theta}_d &= \sqrt{\frac{m_d}{m_s}}, \end{aligned} \quad (4.22)$$

and $\tilde{\varphi} = \psi$ for (4.18), then the result in (4.20) and (4.21) can exactly be reproduced.

Indeed the relation in (4.18) or (4.20), analogous to that in (3.31), defines a triangle in the complex plane (see Fig. 4.1 and Fig. 3.3 for illustration), which has been named as the “light-quark triangle”(LT) in section 3.6. Taking into account the central values of the Cabibbo angle ($\sin \theta_C = |V_{us}| = 0.2205$ [1]) and the light quark mass ratios ($m_s/m_d = 18.8$ and $m_c/m_u = 265$ [10, 11], we can calculate the phase parameter from (4.20) and obtain

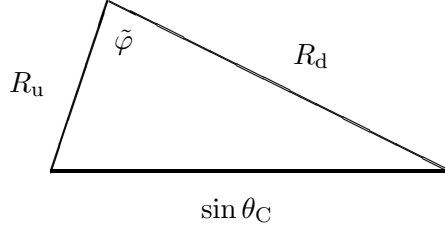


Figure 4.1: The light-quark triangle (LT) in the complex plane.

$\tilde{\varphi} = \psi \approx 79^\circ$. If we allow the mass ratios and θ_C to vary in their ranges given above, then $\tilde{\varphi}$ may vary in the range 38° to 115° . We find that $\tilde{\varphi}$ has a good chance to be around 90° . The case $\tilde{\varphi} \approx 90^\circ$ (i.e., the LT is rectangular) is of special interest, as we shall see later, since it implies that the area of the unitarity triangle of flavor mixing takes its maximum value for the fixed quark mass ratios – in this sense, the CP symmetry of weak interactions would be maximally violated.

It is worth remarking that the quark mass ratios m_d/m_s and m_u/m_c are essentially independent of the renormalization-group effect from the weak scale to a superhigh scale (or vice versa), so is the Cabibbo angle θ_C . As a result the sides and angles of the LT are to a very good degree of accuracy scale-independent. This interesting feature of the light quark sector implies that the prediction for $\tilde{\theta}_u$ and $\tilde{\theta}_d$ from quark mass matrices at any high energy scale (e.g., $\mu \sim 10^{16}$ GeV) can directly be confronted with the low-scale experimental data.

The two symmetry limits discussed above are both not far from the reality, in which the strong hierarchy of quark masses ($m_u \ll m_c \ll m_t$ and $m_d \ll m_s \ll m_b$) has been observed. They will serve as a guide in the subsequent discussions about specific quark mass matrices and their consequences on flavor mixing. In particular one expects that a realistic pattern of 3×3 quark mass matrices should naturally reproduce the 2×2 textures (4.14) and (4.19), respectively, in the chiral limit and in the heavy quark limit. This observation implies that a specific 3×3 texture of the form (4.2) with $E_q = 0$ might be the best candidate for realistic quark mass matrices, which is able to accommodate the experimental data on quark masses, flavor mixing and CP violation. A detailed analysis of this texture will be given later on.

4.3 A Hermitian scheme with four texture zeros

In order to get predictions for the flavor mixing angles and CP violation, we proceed to specify the general Hermitian mass matrices in (4.2) by taking $E_u = E_d = 0$:

$$\begin{aligned} M_u &= \begin{pmatrix} \mathbf{0} & D_u & \mathbf{0} \\ D_u^* & C_u & B_u \\ \mathbf{0} & B_u^* & A_u \end{pmatrix}, \\ M_d &= \begin{pmatrix} \mathbf{0} & D_d & \mathbf{0} \\ D_d^* & C_d & B_d \\ \mathbf{0} & B_d^* & A_d \end{pmatrix}. \end{aligned} \quad (4.23)$$

Then M_u and M_d totally have four texture zeros (here a pair of off-diagonal texture zeros are counted, due to the Hermiticity, as one zero). As remarked above, the texture zeros in (1,3) and (3,1) positions of a general Hermitian mass matrix can always be arranged. Thus the physical constraint is as follows: In the flavor basis in which (1,3) and (3,1) elements of $M_{u,d}$ vanish, the (1,1) element of $M_{u,d}$ vanishes as well. This may be true only at a particular energy scale. The vanishing of the (1,1) element can be viewed as a result of an underlying flavor symmetry, which may either be discrete or continuous. In the literature a number of such possibilities have been discussed (see, e.g., Refs. [47]–[51]). Here we shall not discuss further details in this respect, but concentrate on the phenomenological consequences of M_u and M_d on flavor mixing and CP violation.

Flavor mixing angles Now we start from (4.23) to calculate the flavor mixing matrix. Here again we take $C_q \neq 0$ and define $|B_q|/C_q \equiv r_q$ for each quark sector. The magnitude of r_q is expected to be of $\mathcal{O}(1)$. The parameters A_q , $|B_q|$, C_q and $|D_q|$ of M_q can be expressed in terms of the quark mass eigenvalues and r_q . Applying such results to the general formulas listed in (4.9) and (4.10), we get three mixing angles of V as follows:

$$\begin{aligned}\tan \theta_u &= \sqrt{\frac{m_u}{m_c}} (1 + \Delta_u) , \\ \tan \theta_d &= \sqrt{\frac{m_d}{m_s}} (1 + \Delta_d) ,\end{aligned}\tag{4.24}$$

and

$$\sin \theta = \left| r_d \frac{m_s}{m_b} (1 - \delta_d) - r_u \frac{m_c}{m_t} (1 - \delta_u) e^{i\phi_2} \right| ,\tag{4.25}$$

where the next-to-leading order corrections read

$$\begin{aligned}\Delta_u &= \sqrt{\frac{m_c m_d}{m_u m_s}} \frac{m_s}{m_b} \left| \text{Re} \left[e^{i\phi_1} - \frac{r_u}{r_d} \cdot \frac{m_c m_b}{m_t m_s} e^{i(\phi_1 + \phi_2)} \right]^{-1} \right| , \\ \Delta_d &= \sqrt{\frac{m_u m_s}{m_c m_d}} \frac{m_c}{m_t} \left| \text{Re} \left[e^{i\phi_1} - \frac{r_d}{r_u} \cdot \frac{m_t m_s}{m_c m_b} e^{i(\phi_1 + \phi_2)} \right]^{-1} \right| ;\end{aligned}\tag{4.26}$$

as well as

$$\begin{aligned}\delta_u &= \frac{m_u}{m_c} + \left(1 + r_u^2 \right) \frac{m_c}{m_t} , \\ \delta_d &= \frac{m_d}{m_s} + \left(1 + r_d^2 \right) \frac{m_s}{m_b} .\end{aligned}\tag{4.27}$$

Clearly the result for $\hat{\delta}_{u,d}$ in (4.16) can be reproduced from $\delta_{u,d}$ in (4.27) if one takes the chiral limit $m_u \rightarrow 0$, $m_d \rightarrow 0$. From (4.25) we also observe that the phase ϕ_2 is only associated with the small quantity m_c/m_t in $\sin \theta$. To get the relationship between φ and ϕ_1 or ϕ_2 , we first calculate $|V_{us}|$ from quark mass matrices by use of (4.12). We obtain

$$|V_{us}| = \left(1 - \frac{1}{2} \frac{m_u}{m_c} - \frac{1}{2} \frac{m_d}{m_s} \right) \left| \sqrt{\frac{m_d}{m_s}} - \sqrt{\frac{m_u}{m_c}} e^{i\phi_1} \right| ,\tag{4.28}$$

in the next-to-leading order approximation. Note that this result can also be achieved from (4.20) and (4.21), which were obtained in the heavy quark limit. Confronting (4.28)

with current data on $|V_{us}|$ leads to the result $\phi_1 \sim 90^\circ$, as we have pointed out before. Therefore $\cos \phi_1$ is expected to be a small quantity. Then we use (4.11) together with (4.24) and (4.28) to calculate $\cos \varphi$. In the same order approximation, we arrive at

$$\cos \varphi = \sqrt{\frac{m_u m_s}{m_c m_d}} \Delta_u + \sqrt{\frac{m_c m_d}{m_u m_s}} \Delta_d + (1 - \Delta_u - \Delta_d) \cos \phi_1 . \quad (4.29)$$

The contribution of ϕ_2 to φ is substantially suppressed at this level of accuracy.

For simplicity, we proceed by taking $r_u = r_d \equiv r$, which holds in some aforementioned models with natural flavor symmetries [50]. Then $\sin \theta$ becomes proportional to a universal parameter $|r|$. In view of the fact $m_s/m_b \sim \mathcal{O}(10) m_c/m_t$, we find that the result in (4.26) can be simplified as

$$\begin{aligned} \Delta_u &= \sqrt{\frac{m_c m_d}{m_u m_s}} \frac{m_s}{m_b} \cos \phi_1 , \\ \Delta_d &= 0 . \end{aligned} \quad (4.30)$$

Also the relation between φ and ϕ_1 in (4.29) is simplified to

$$\cos \varphi = \left(1 + \frac{m_s}{m_b} \right) \cos \phi_1 . \quad (4.31)$$

As $m_s/m_b \sim 4\%$, it becomes apparent that $\varphi \approx \phi_1$ is a good approximation. With these results one may further calculate the rephasing-invariant CP -violating parameter \mathcal{J} defined in (3.11). We confirm that the magnitude of \mathcal{J} is dominated by the $\sin \phi_1$ term and receives one-order smaller corrections from the $\sin(\phi_1 \pm \phi_2)$ terms. As a result,

$$\mathcal{J} \approx |r|^2 \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{m_d}{m_s}} \left(\frac{m_s}{m_b} \right)^2 \sin \phi_1 \quad (4.32)$$

holds to a good degree of accuracy. Clearly $\mathcal{J} \sim \mathcal{O}(10^{-5}) \times \sin \phi_1$ with $\sin \phi_1 \sim 1$ is favored by the current experimental data.

The result of \mathcal{J} in (4.32) might give one an impression that CP violation is absent if either m_u or m_d vanishes. This is not exactly true, however. If we set $m_u = 0$, \mathcal{J} is not zero, but it becomes dependent on $\sin \phi_2$ with a factor which is about two orders of magnitude smaller (i.e., of order 10^{-7}):

$$\mathcal{J} \approx |r|^2 \frac{m_c}{m_t} \cdot \frac{m_d}{m_s} \left(\frac{m_s}{m_b} \right)^2 \sin \phi_2 . \quad (4.33)$$

Certainly this possibility has been ruled out by the present experimental data (i.e., both m_u and m_d must be different from zero).

Note also that the texture of M_u and M_d in (4.23) predicts

$$\begin{aligned} \tan \theta_u &= \left| \frac{V_{ub}}{V_{cb}} \right| = \sqrt{\frac{m_u}{m_c}} (1 + \Delta_u) , \\ \tan \theta_d &= \left| \frac{V_{td}}{V_{ts}} \right| = \sqrt{\frac{m_d}{m_s}} (1 + \Delta_d) . \end{aligned} \quad (4.34)$$

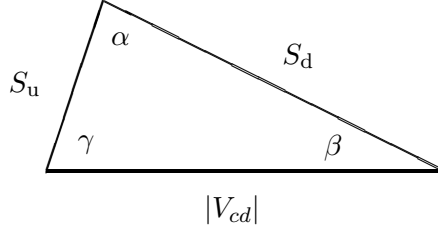


Figure 4.2: The rescaled unitarity triangle (UT) in the complex plane.

In B -meson physics, $|V_{ub}/V_{cb}|$ can be determined from the ratio of the decay rate of $B \rightarrow (\pi, \rho) l \nu_l$ to that of $B \rightarrow D^* l \nu_l$; and $|V_{td}/V_{ts}|$ can be extracted from the ratio of the rate of $B_d^0 - \bar{B}_d^0$ mixing to that of $B_s^0 - \bar{B}_s^0$ mixing [42].

The unitarity triangle We are now in a position to calculate the unitarity triangle (UT) of quark flavor mixing defined by the orthogonality relation

$$V_{ub}^* V_{ud} + V_{cb}^* V_{cd} + V_{tb}^* V_{td} = 0, \quad (4.35)$$

i.e., the triangle Δ_s in Fig. 3.1. The three inner angles of this triangle are denoted as α , β and γ in (3.20). Three sides of Δ_s can be rescaled by V_{cb}^* (see Fig. 3.3(b) and Fig. 4.2 for illustration). The resultant triangle reads

$$|V_{cd}| = |S_d - S_u e^{-i\alpha}|, \quad (4.36)$$

where $S_u = |V_{ub}V_{ud}/V_{cb}|$ and $S_d = |V_{tb}V_{td}/V_{cb}|$. After some calculations we obtain S_u , S_d and α from M_u and M_d in the next-to-leading order approximation [8]:

$$\begin{aligned} S_u &= \sqrt{\frac{m_u}{m_c}} \left(1 - \frac{1}{2} \frac{m_u}{m_c} - \frac{1}{2} \frac{m_d}{m_s} + \sqrt{\frac{m_c m_d}{m_u m_s}} \frac{m_s}{m_b} \cos \phi_1 + \sqrt{\frac{m_u m_d}{m_c m_s}} \cos \phi_1 \right), \\ S_d &= \sqrt{\frac{m_d}{m_s}} \left(1 + \frac{1}{2} \frac{m_u}{m_c} - \frac{1}{2} \frac{m_d}{m_s} \right); \end{aligned} \quad (4.37)$$

and

$$\sin \alpha = \left(1 - \sqrt{\frac{m_u m_d}{m_c m_s}} \cos \phi_1 \right) \sin \phi_1. \quad (4.38)$$

A comparison of the rescaled UT in Fig. 4.2 with the LT in Fig. 4.1, which is obtained in the heavy quark limit, is interesting. We find

$$\begin{aligned} \frac{S_u - R_u}{R_u} &= \left(1 + \frac{m_c m_s}{m_u m_b} \right) \sqrt{\frac{m_u m_d}{m_c m_s}} \cos \tilde{\varphi}, \\ \frac{S_d - R_d}{R_d} &= \frac{m_u}{m_c}, \\ \frac{\sin \alpha - \sin \tilde{\varphi}}{\sin \tilde{\varphi}} &= -\sqrt{\frac{m_u m_d}{m_c m_s}} \cos \tilde{\varphi}, \end{aligned} \quad (4.39)$$

which are of order $15\% \cos \tilde{\varphi}$, 0.4% and $1.4\% \cos \tilde{\varphi}$, respectively. Obviously $R_d \approx S_d$ is an excellent approximation, and $\alpha \approx \tilde{\varphi} \approx \varphi$ is a good approximation. As φ (or $\tilde{\varphi}$) is

expected to be close to 90° , $R_u \approx S_u$ should also be accurate enough in the next-to-leading order estimation. Therefore the light-quark triangle is essentially *congruent with* the rescaled unitarity triangle! This result has two straightforward implications: first, CP violation is an effect arising primarily from the light quark sector; second, the CP -violating observables (α , β , γ) can be predicted in terms of the light quark masses and the phase difference between up and down mass matrices [47]. If we use the value of $|V_{cd}|$, which is expected to be equal to $|V_{us}|$ within the 0.1% error bar [37], then all three angles of the unitarity triangle can be calculated in terms of m_u/m_c , m_d/m_s and $|V_{cd}|$ to a good degree of accuracy.

The three inner angles of the UT (α , β and γ) will be determined at B -meson factories, e.g., from the CP asymmetries in $B_d \rightarrow \pi^+\pi^-$, $B_d \rightarrow J/\psi K_S$ and $B_u^\pm \rightarrow (D^0, \bar{D}^0) + K^{(*)\pm}$ decays [52]. The characteristic measurable quantities are $\sin(2\alpha)$, $\sin(2\beta)$ and $\sin^2 \gamma$, respectively. For the purpose of illustration, we typically take $|V_{us}| = |V_{cd}| = 0.22$, $m_u/m_c = 0.0056$, $m_d/m_s = 0.045$ and $m_s/m_b = 0.033$ to calculate these three CP -violating parameters from the LT and from the rescaled UT separately. Both approaches lead to $\alpha \approx 90^\circ$, $\beta \approx 20^\circ$ and $\gamma \approx 70^\circ$, which are in good agreement with the results obtained from the standard analysis of current data on $|V_{ub}/V_{cb}|$, ϵ_K , B_d^0 - \bar{B}_d^0 mixing and B_s^0 - \bar{B}_s^0 mixing [46]. Note that among three CP -violating observables only $\sin(2\beta)$ is remarkably sensitive to the value of m_u/m_c , which involves quite large uncertainty (e.g., $\sin(2\beta)$ may change from 0.4 to 0.8 if m_u/m_c varies in the range 0.002 to 0.01). For this reason we emphasize again that the numbers given above can only serve as an illustration. A more reliable determination of the quark mass values is crucial, in order to test the patterns of quark mass matrices in a numerically decisive way [53].

It is also worth mentioning that the result $\tan \theta_d = \sqrt{m_d/m_s}$ is particularly interesting for the mixing rates of B_d^0 - \bar{B}_d^0 and B_s^0 - \bar{B}_s^0 systems, measured by x_d and x_s respectively. The ratio x_s/x_d amounts to $|V_{ts}/V_{td}|^2 = \tan^{-2} \theta_d$ multiplied by a factor $\chi_{\text{su}(3)} = 1.45 \pm 0.13$, which reflects the $\text{SU}(3)_{\text{flavor}}$ symmetry breaking effects [54]. As $x_d = 0.723 \pm 0.032$ has been well determined [1], the prediction for the value of x_s is

$$x_s = x_d \chi_{\text{su}(3)} \frac{m_s}{m_d} = 19.8 \pm 3.5, \quad (4.40)$$

where $m_s/m_d = 18.9 \pm 0.8$, obtained from the chiral perturbation theory [11], has been used. This result is certainly consistent with the present experimental bound on x_s , i.e., $x_s > 14.0$ at the 95% confidence level [1]. A measurement of $x_s \sim 20$ may be realized at the forthcoming HERA- B and LHC- B experiments.

Further discussions Finally let us make some further remarks on the quark mass matrices (4.23), its phenomenological hints and its theoretical prospects.

(1) Naively one might not expect any prediction from the four-texture-zero mass matrices in (4.23), since they totally consist of ten free parameters (two of them are the phase differences between M_u and M_d). This is not true, however, as we have seen. We find that two predictions, $\tan \theta_u \approx \sqrt{m_u/m_c}$ and $\tan \theta_d \approx \sqrt{m_d/m_s}$, can be obtained in the leading order approximation. In some cases the latter may even hold in the next-to-leading order approximation, as shown in (4.30). Note again that these two relations, as a consequence of the hierarchy and the texture zeros of quark mass matrices, are essentially independent

of the renormalization-group effects. This interesting scale-independent feature has also manifested itself in the LT and the rescaled UT, as well as their inner angles (α, β, γ) .

(2) It remains to be seen whether the interesting possibility $\varphi \approx \phi_1 \approx 90^\circ$, indicated by current data of quark masses and flavor mixing, could arise from an underlying flavor symmetry or a dynamical symmetry breaking scheme. Some speculations about this problem have been made (see, e.g., Refs. [8, 47] and Refs. [55, 56]). However, no final conclusion has been reached thus far. It is remarkable, nevertheless, that we have at least observed a useful relation between the area of the UT (\mathcal{A}_{UT}) and that of the LT (\mathcal{A}_{LT}) to a good degree of accuracy [8]:

$$\mathcal{A}_{\text{UT}} \approx |V_{cb}|^2 \mathcal{A}_{\text{LT}} \approx \sin^2 \theta \mathcal{A}_{\text{LT}}. \quad (4.41)$$

Since $\mathcal{A}_{\text{UT}} = \mathcal{J}/2$ measures the magnitude of CP violation in the standard model, we conclude that CP violation is primarily linked to the light quark sector. This is a natural consequence of the strong hierarchy between the heavy and light quark masses, which is on the other hand responsible for the smallness of \mathcal{J} or \mathcal{A}_{UT} .

(3) Is it possible to derive the quark mass matrix (4.23) in an underlying theoretical framework? To answer this question we first specify the hierarchical structure of M_q in terms of the mixing angle θ_q (for $q = d$ or s). Adopting the radiant unit for the mixing angles (i.e., $\theta_u \approx 0.085$, $\theta_d \approx 0.204$ and $\theta \approx 0.040$), we have

$$\begin{aligned} \frac{m_u}{m_c} &\sim \frac{m_c}{m_t} \sim \theta_u^2, \\ \frac{m_d}{m_s} &\sim \frac{m_s}{m_b} \sim \theta_d^2. \end{aligned} \quad (4.42)$$

Then the mass matrices M_u and M_d , which have the mass scales m_t and m_b respectively, take the following *parallel* hierarchies:

$$\begin{aligned} M_u &\sim m_t \begin{pmatrix} \mathbf{0} & \theta_u^3 & \mathbf{0} \\ \theta_u^3 & \theta_u^2 & \theta_u^2 \\ \mathbf{0} & \theta_u^2 & \mathbf{1} \end{pmatrix}, \\ M_d &\sim m_b \begin{pmatrix} \mathbf{0} & \theta_d^3 & \mathbf{0} \\ \theta_d^3 & \theta_d^2 & \theta_d^2 \\ \mathbf{0} & \theta_d^2 & \mathbf{1} \end{pmatrix}, \end{aligned} \quad (4.43)$$

where the relevant complex phases have been neglected. Clearly all three flavor mixing angles can properly be reproduced from (4.43), once one takes $\theta \approx \theta_d^2 \gg \theta_u^2$ into account. The CP -violating phase φ in V comes essentially from the phase difference between the θ_u^3 and θ_d^3 terms.

Of course θ_u and θ_d , which are more fundamental than the Cabibbo angle θ_C in our point of view, denote perturbative corrections to the rank-one limits of M_u and M_d respectively. They are responsible for the generation of light quark masses as well as the flavor mixing. They might also be responsible for CP violation in a specific theoretical framework (e.g., a pure real θ_u and a pure imaginary θ_d will lead to a phase difference of about 90° between M_u and M_d , which is just the source of CP violation favored by current data). The small parameter θ_q could get its physical meaning in the Yukawa coupling of

an underlying superstring theory: $\theta_q = \langle \Theta_q \rangle / \Omega_q$, where $\langle \Theta_q \rangle$ denotes the vacuum expectation value of the singlet field Θ_q , and Ω_q represents the unification (or string) mass scale which governs higher dimension operators (see, e.g., Refs. [6, 57, 58]). The quark mass matrices of the form (4.43) could then be obtained by introducing an extra (horizontal) $U(1)$ gauge symmetry or assigning the matter fields appropriately.

It is concluded that the four texture zeros and parallel hierarchies of up and down quark mass matrices do imply specific symmetries, perhaps at a superhigh scale, and have instructive consequences on flavor mixing and CP -violating phenomena. The new parametrization of the flavor mixing matrix that we advocated in (3.29) is particularly useful in studying the quark mass generation, flavor mixing and CP violation.

4.4 Hermitian schemes with five texture zeros

The quark mass matrices M_u and M_d given in (4.23) have parallel structures with four texture zeros. Giving up the parallelism between the structures of M_u and M_d , Ramond, Roberts and Ross (RRR) have found that there exist five phenomenologically allowed patterns of Hermitian quark mass matrices – each of them has five texture zeros [59], as listed in Table 4.1. The RRR patterns I, II and IV can be formally regarded as a special case of the four-texture-zero pattern (4.23), with $B_u = 0$, $C_u = 0$ and $B_d = 0$, respectively. Note that M_u of the RRR pattern III or V has nonvanishing (1,3) and (3,1) elements, therefore these two patterns are essentially different from the mass matrices prescribed in (4.2) or (4.23). As a comparison, here we make some brief comments on consequences of the RRR patterns on the flavor mixing angles θ , θ_u and θ_d .

(a) For the RRR pattern I, the magnitude of $\sin \theta$ is governed by the (2,3) and (3,2) elements of M_d . Therefore $|B_d| \sim |C_d|$ is expected, to result in $\sin \theta \sim m_s/m_b$. In the leading order approximation, $\tan \theta_u = \sqrt{m_u/m_c}$ and $\tan \theta_d = \sqrt{m_d/m_s}$ hold. The next-to-leading order corrections to these two quantities are almost indistinguishable from those obtained in (4.26) and (4.30) for the four-texture-zero ansatz.

(b) The mass matrix M_u of the RRR pattern II takes the well-known form suggested originally in Ref. [4]. To reproduce the experimental value of $\sin \theta$, the possibility $|B_d| \gg |C_d|$ has to be abandoned and the condition $|B_d| \sim |C_d|$ is required. However, significant cancellation between the term proportional to $\sqrt{m_c/m_t}$ (from M_u) and that proportional to m_s/m_b (from M_d) in $\sin \theta$ may take place, if the phase difference between B_u and B_d is vanishing or very small. The leading order results $\tan \theta_u = \sqrt{m_u/m_c}$ and $\tan \theta_d = \sqrt{m_d/m_s}$ can still be obtained here, but their next-to-leading order corrections may deviate somehow from those obtained in section 4.3.

(c) From the RRR pattern IV one can arrive at $\sin \theta \sim \sqrt{m_c/m_t}$ with the necessary condition $|C_u| \ll |B_u|$, since the (2,3) and (3,2) elements of M_d vanish. The results for $\tan \theta_u$ and $\tan \theta_d$ are similar to those obtained from the RRR pattern I.

(d) The nonvanishing (1,3) and (3,1) elements of M_u in the RRR pattern III make its prediction for the mixing angles θ_u and θ_d quite different from all patterns discussed above. Analytically one can find $\tan \theta_u \sim (m_b/m_s)\sqrt{m_u/m_t}$, while $\tan \theta_d$ is a complicated combination of the terms $\sqrt{m_d/m_s}$ and $(m_b/m_s)\sqrt{m_u/m_t}$ with a relative phase. In addition, $\sin \theta \sim m_s/m_b$ holds under the condition $|B_d| \sim |C_d|$, similar to the RRR pattern I.

Table 4.1: Five RRR patterns of Hermitian quark mass matrices.

| Pattern | M_u | M_d |
|---------|--|--|
| I | $\begin{pmatrix} \mathbf{0} & D_u & \mathbf{0} \\ D_u^* & C_u & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_u \end{pmatrix}$ | $\begin{pmatrix} \mathbf{0} & D_d & \mathbf{0} \\ D_d^* & C_d & B_d \\ \mathbf{0} & B_d^* & A_d \end{pmatrix}$ |
| II | $\begin{pmatrix} \mathbf{0} & D_u & \mathbf{0} \\ D_u^* & \mathbf{0} & B_u \\ \mathbf{0} & B_u^* & A_u \end{pmatrix}$ | $\begin{pmatrix} \mathbf{0} & D_d & \mathbf{0} \\ D_d^* & C_d & B_d \\ \mathbf{0} & B_d^* & A_d \end{pmatrix}$ |
| III | $\begin{pmatrix} \mathbf{0} & \mathbf{0} & F_u \\ \mathbf{0} & C_u & \mathbf{0} \\ F_u^* & \mathbf{0} & A_u \end{pmatrix}$ | $\begin{pmatrix} \mathbf{0} & D_d & \mathbf{0} \\ D_d^* & C_d & B_d \\ \mathbf{0} & B_d^* & A_d \end{pmatrix}$ |
| IV | $\begin{pmatrix} \mathbf{0} & D_u & \mathbf{0} \\ D_u^* & C_u & B_u \\ \mathbf{0} & B_u^* & A_u \end{pmatrix}$ | $\begin{pmatrix} \mathbf{0} & D_d & \mathbf{0} \\ D_d^* & C_d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_d \end{pmatrix}$ |
| V | $\begin{pmatrix} \mathbf{0} & \mathbf{0} & F_u \\ \mathbf{0} & C_u & B_u \\ F_u^* & B_u^* & A_u \end{pmatrix}$ | $\begin{pmatrix} \mathbf{0} & D_d & \mathbf{0} \\ D_d^* & C_d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_d \end{pmatrix}$ |

(e) For the RRR pattern V, the necessary condition $|B_u| \gg |C_u|$ is required in order to reproduce $\sin \theta \sim \sqrt{m_c/m_t}$. Here again the nonvanishing (1,3) and (3,1) elements of M_u result in very complicated expressions for $\tan \theta_u$ and $\tan \theta_d$ (even more complicated than those in the RRR pattern III [58]).

For reasons of naturalness and simplicity, we argue that the RRR patterns III and V are unlikely to be good candidates for the quark mass matrices in an underlying theory of fermion mass generation. In analogy with (4.43), we illustrate different magnitudes of the elements of five RRR-type mass matrices approximately in terms of two basic flavor mixing parameters θ_u and θ_d . The results are listed in Table 4.2, where the relevant complex phases of M_u and M_d are neglected. It is clear that the hierarchy of either (up or down) mass matrix can be governed by a single small parameter, whose physical mixing is apparently related to the smallness of the flavor mixing angles.

A numerical illustration of the consequences of five RRR ansätze on flavor mixing and CP violation has been given in Ref. [59]. It is worth remarking that with the same input values of quark mass ratios only one or two of them can be in good agreement with current experimental data. In other words, the five textures cannot simultaneously survive.

4.5 Mass matrices from flavor democracy breaking

Towards an understanding of the hierarchy of quark masses and flavor mixing angles, it has been speculated by a number of authors that the realistic quark mass matrices like that in (4.23) might arise from the flavor democracy symmetry and its explicit breaking [50].

Table 4.2: Approximate forms of real RRR-type quark mass matrices.

| Pattern | M_u | M_d |
|---------|---|---|
| I | $m_t \begin{pmatrix} \mathbf{0} & \theta_u^3 & \mathbf{0} \\ \theta_u^3 & \theta_u^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix}$ | $m_b \begin{pmatrix} \mathbf{0} & \theta_d^3 & \mathbf{0} \\ \theta_d^2 & \theta_d^2 & \theta_d^2 \\ \mathbf{0} & \theta_d^2 & 1 \end{pmatrix}$ |
| II | $m_t \begin{pmatrix} \mathbf{0} & \theta_u^3 & \mathbf{0} \\ \theta_u^3 & \mathbf{0} & \theta_u \\ \mathbf{0} & \theta_u & 1 \end{pmatrix}$ | $m_b \begin{pmatrix} \mathbf{0} & \theta_d^3 & \mathbf{0} \\ \theta_d^3 & \theta_d^2 & \theta_d^2 \\ \mathbf{0} & \theta_d^2 & 1 \end{pmatrix}$ |
| III | $m_t \begin{pmatrix} \mathbf{0} & \mathbf{0} & \theta_u^2 \\ \mathbf{0} & \theta_u^2 & \mathbf{0} \\ \theta_u^2 & \mathbf{0} & 1 \end{pmatrix}$ | $m_b \begin{pmatrix} \mathbf{0} & \theta_d^3 & \mathbf{0} \\ \theta_d^3 & \theta_d^2 & \theta_d^2 \\ \mathbf{0} & \theta_d^2 & 1 \end{pmatrix}$ |
| IV | $m_t \begin{pmatrix} \mathbf{0} & \theta_u^3 & \mathbf{0} \\ \theta_u^3 & \theta_u^2 & \theta_u \\ \mathbf{0} & \theta_u & 1 \end{pmatrix}$ | $m_b \begin{pmatrix} \mathbf{0} & \theta_d^3 & \mathbf{0} \\ \theta_d^3 & \theta_d^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix}$ |
| V | $m_t \begin{pmatrix} \mathbf{0} & \mathbf{0} & \theta_u^2 \\ \mathbf{0} & \theta_u^2 & \theta_u \\ \theta_u^2 & \theta_u & 1 \end{pmatrix}$ | $m_b \begin{pmatrix} \mathbf{0} & \theta_d^3 & \mathbf{0} \\ \theta_d^3 & \theta_d^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix}$ |

Let us describe the basic idea of this approach and give a simple example for illustration.

Under exact $S(3)_L \times S(3)_R$ symmetry (i.e., the flavor democracy), the mass spectrum for either up or down quark sector consists of only two levels: one is of two-fold degeneracy with vanishing mass eigenvalues, and the other is nondegenerate and massive. The mass matrix in this “democratic” basis is of rank one and takes the form

$$M_q^{(0)} = \frac{\chi_q}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad (4.44)$$

with $\chi_u = m_t$ and $\chi_d = m_b$. Through the orthogonal transformation $U_0 M_q^{(0)} U_0^\dagger$, where

$$U_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad (4.45)$$

one arrives at another rank-one matrix in the “hierarchical” basis:

$$M_q^{(H)} = \chi_q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (4.46)$$

like that in (2.10). Clearly quarks of the first two families are degenerate and massless. The mass matrices $M_q^{(H)}$ and $M_q^{(0)}$ have equivalent physical significance and either of them

can be used as a starting symmetry limit to construct the full quark mass matrix. However, we should like to remark that the democratic pattern (4.44) plays an important role in some other aspects of physics, where the mass-gap phenomena have also been observed. For example, in the BCS theory of superconductivity, the energy gap is related to a democratic matrix in the Hilbert space of the Cooper pairs [60]; the pairing force in nuclear physics, which is introduced to explain large mass gaps in nuclear energy levels, has the property that the associated Hamiltonian in the space of nucleon pairs possesses equal mass elements, i.e., it has the structure like $M_q^{(0)}$ [61]; the mass pattern of the (π^0, η, η') pseudoscalar mesons in QCD can be described by $M_q^{(0)}$ in the chiral limit $m_u = m_d = 0$, where π^0 and η are massless Goldstone bosons but η' has a nonvanishing mass as a consequence of the gluon anomaly [62]. Such examples, together with the observed hierarchy in quark and charged lepton mass spectra, might imply that the mass-gap phenomena in physics come in general from analogous dynamical mechanisms with the underlying democracy symmetry.

The $S(3)_L \times S(3)_R$ symmetry of $M_q^{(0)}$ need be broken to the $S(2)_L \times S(2)_R$ symmetry, in order to generate masses of the second-family quarks. For this purpose we consider the following (real) symmetry-breaking correction to $M_q^{(0)}$:

$$\Delta M_q^{(1)} = \frac{\chi_q}{3} \begin{pmatrix} 0 & 0 & \delta_q \\ 0 & 0 & \delta_q \\ \delta_q & \delta_q & \varepsilon_q \end{pmatrix}, \quad (4.47)$$

where $|\delta_q| \ll 1$ and $|\varepsilon_q| \ll 1$. Diagonalizing $M_q^{(0)} + \Delta M_q^{(1)}$ through the orthogonal transformation which depends on a single rotation angle, one can obtain quark mass eigenvalues of the second and third families. Explicitly,

$$\begin{aligned} m_c &= \frac{\chi_u}{6} \left[3 + \varepsilon_u - \sqrt{(1 - \varepsilon_u)^2 + 8(1 + \delta_u)^2} \right], \\ m_t &= \frac{\chi_u}{6} \left[3 + \varepsilon_u + \sqrt{(1 - \varepsilon_u)^2 + 8(1 + \delta_u)^2} \right]; \end{aligned} \quad (4.48)$$

and m_s, m_b can similarly be given in terms of ε_d and δ_d . The flavor mixing angle between the second and third quark families, denoted as $\hat{\theta}$, reads as

$$\begin{aligned} \sin \hat{\theta} &= \frac{\sqrt{2}}{9} \left| (\varepsilon_u - \varepsilon_d) \left[1 + \frac{1}{9} (\varepsilon_u + \varepsilon_d) \right] + (\delta_u - \delta_d) \left[1 - \frac{8}{9} (\delta_u + \delta_d) \right] \right. \\ &\quad \left. - \frac{7}{9} (\varepsilon_u \delta_u - \varepsilon_d \delta_d) \right| \end{aligned} \quad (4.49)$$

in the next-to-leading order approximation. We see that the magnitude of this mixing parameter (i.e., $|V_{cb}|$ or $|V_{ts}|$) is dominated by the linear difference between ε_u and ε_d and (or) that between δ_u and δ_d . Two instructive examples are in order:

(a) $\delta_q = 0$ yielding

$$\sin \hat{\theta} = \frac{1}{\sqrt{2}} \left(\frac{m_s}{m_b} - \frac{m_c}{m_t} \right) \left[1 + \frac{3}{2} \left(\frac{m_s}{m_b} + \frac{m_c}{m_t} \right) \right]; \quad (4.50)$$

(b) $\delta_q = \varepsilon_q$ yielding

$$\sin \hat{\theta} = \sqrt{2} \left(\frac{m_s}{m_b} - \frac{m_c}{m_t} \right) \left[1 + 3 \left(\frac{m_s}{m_b} + \frac{m_c}{m_t} \right) \right]. \quad (4.51)$$

These two cases have been discussed in Refs. [63] and [64], respectively. The latter is more favored by current data on the magnitude of V_{cb} .

To generate masses of the first-family quarks and CP violation, a further (complex) perturbation need be introduced to $M_q^{(0)}$. Then the $S(3)_L \times S(3)_R$ symmetry of $M_q^{(0)}$ will be broken to $S(1)_L \times S(1)_R$. A simple symmetry-breaking pattern of this category is

$$\Delta M_q^{(2)} = \frac{\chi_q}{3} \left[\sigma_q \cos \phi_q \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} + i \sigma_q \sin \phi_q \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \right], \quad (4.52)$$

where $|\sigma_q| \ll 1$. Transforming $M_q^{(0)} + \Delta M_q^{(1)} + \Delta M_q^{(2)}$ into the heavy basis, we obtain (for the case $\delta_q = \varepsilon_q$) the following mass matrices:

$$M_q = \chi_q \begin{pmatrix} \mathbf{0} & \frac{1}{\sqrt{3}} \sigma_q e^{+i\phi_q} & \mathbf{0} \\ \frac{1}{\sqrt{3}} \sigma_q e^{-i\phi_q} & -\frac{2}{9} \varepsilon_q & -\frac{2\sqrt{2}}{9} \varepsilon_q \\ \mathbf{0} & -\frac{2\sqrt{2}}{9} \varepsilon_q & 1 + \frac{5}{9} \varepsilon_q \end{pmatrix}. \quad (4.53)$$

We see that M_u and M_d totally have four texture zeros, just like the more general pattern given in (4.23). This implies that consequences of the mass matrices in (4.53) on flavor mixing and CP violation can essentially be obtained with the help of the relevant formulas derived from (4.23) in section 4.3; i.e., the former may result from the latter by taking $\phi_1 = \phi_u - \phi_d$, $\phi_2 = 0$ and $|r_u| = |r_d| = \sqrt{2}$ [64]. We find that the results for θ_u , θ_d and angles of the unitarity triangles are almost insensitive to the explicit values of ϕ_2 and $|r_u| = |r_d|$, which mainly affect the mixing angle θ . The point is simply that the hierarchy of quark masses and the texture of quark mass matrices are enough to determine, at least partly, some features of the flavor mixing. In this sense we emphasize again that the scheme (4.23) definitely has some predictive power for the flavor mixing angles, although it totally involves ten parameters.

Of course there are many other possibilities to break the symmetry of flavor democracy for $M_q^{(0)}$. The corresponding forms of quark mass matrices in the hierarchical basis can be obtained in a straightforward way (see Table 4.3, in which only real perturbations to $M_q^{(0)}$ are considered, for illustration). Conversely one may transform a general (Hermitian) mass matrix from the hierarchical basis to the democratic basis, as also shown by the translation “dictionary” in Table 4.3.

4.6 Non-Hermitian textures of mass matrices

The textures of quark mass matrices discussed above all possess the Hermitian feature. In general, however, hermiticity is not a necessary constraint on the up- or down-type mass matrix, or both of them, for any theory of fermion mass generation. Some attention has

Table 4.3: Translations of quark or lepton mass matrices from the “democratic” basis (M_D) into the “hierarchical” basis (M_H), or vice versa.

| Democratic pattern M_D | | Hierarchical pattern M_H | Flavor symmetry of M_D |
|---|---------------|--|--------------------------|
| $c \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ | \Rightarrow | $3c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $S(3)_L \times S(3)_R$ |
| $c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}$ | \Rightarrow | $\frac{c\epsilon}{3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & -\sqrt{2} \\ 0 & -\sqrt{2} & 1 \end{pmatrix}$ | $S(2)_L \times S(2)_R$ |
| $c \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | \Rightarrow | $\frac{c\epsilon}{6} \begin{pmatrix} 3 & -\sqrt{3} & -\sqrt{6} \\ -\sqrt{3} & 1 & \sqrt{2} \\ -\sqrt{6} & \sqrt{2} & 2 \end{pmatrix}$ | $S(2)_L \times S(2)_R$ |
| $c \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | \Rightarrow | $\frac{c\epsilon}{6} \begin{pmatrix} 3 & \sqrt{3} & \sqrt{6} \\ \sqrt{3} & 1 & \sqrt{2} \\ \sqrt{6} & \sqrt{2} & 2 \end{pmatrix}$ | $S(2)_L \times S(2)_R$ |
| $c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & \epsilon & 0 \end{pmatrix}$ | \Rightarrow | $\frac{c\epsilon}{3\sqrt{2}} \begin{pmatrix} 0 & \sqrt{6} & -\sqrt{3} \\ \sqrt{6} & -2\sqrt{2} & -1 \\ -\sqrt{3} & -1 & 2\sqrt{2} \end{pmatrix}$ | $S(1)_L \times S(1)_R$ |
| $c \begin{pmatrix} 0 & 0 & \epsilon \\ 0 & 0 & 0 \\ \epsilon & 0 & 0 \end{pmatrix}$ | \Rightarrow | $\frac{c\epsilon}{3\sqrt{2}} \begin{pmatrix} 0 & -\sqrt{6} & \sqrt{3} \\ -\sqrt{6} & -2\sqrt{2} & -1 \\ \sqrt{3} & -1 & 2\sqrt{2} \end{pmatrix}$ | $S(1)_L \times S(1)_R$ |
| $c \begin{pmatrix} 0 & \epsilon & 0 \\ \epsilon & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | \Rightarrow | $\frac{c\epsilon}{3} \begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & \sqrt{2} \\ 0 & \sqrt{2} & 2 \end{pmatrix}$ | $S(1)_L \times S(1)_R$ |
| Hierarchical pattern M_H | | Democratic pattern M_D | Flavor symmetry of M_D |
| $A \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | \Rightarrow | $\frac{A}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ | $S(3)_L \times S(3)_R$ |
| $B \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ | \Rightarrow | $\frac{B}{3\sqrt{2}} \begin{pmatrix} 2 & 2 & -1 \\ 2 & 2 & -1 \\ -1 & -1 & -4 \end{pmatrix}$ | $S(2)_L \times S(2)_R$ |
| $C \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | \Rightarrow | $\frac{C}{6} \begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{pmatrix}$ | $S(2)_L \times S(2)_R$ |
| $D \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | \Rightarrow | $\frac{D}{\sqrt{3}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ | $S(1)_L \times S(1)_R$ |
| $E \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | \Rightarrow | $\frac{E}{2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $S(1)_L \times S(1)_R$ |
| $F \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ | \Rightarrow | $\frac{F}{\sqrt{6}} \begin{pmatrix} 2 & 0 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & 0 \end{pmatrix}$ | $S(1)_L \times S(1)_R$ |

recently been paid to the non-Hermitian schemes of quark mass matrices. In the following we illustrate three typical ansätze of this nature.

Nearest-neighbor mixing pattern In the standard electroweak model or its extensions which do not have flavor-changing right-handed currents, it is always possible to find a specific flavor basis in which the arbitrary 3×3 quark mass matrices take the nearest-neighbor mixing form [65]:

$$\begin{aligned} M_u &= \begin{pmatrix} \mathbf{0} & x_u & \mathbf{0} \\ x'_u & \mathbf{0} & y_u \\ \mathbf{0} & y'_u & z_u \end{pmatrix}, \\ M_d &= \begin{pmatrix} \mathbf{0} & x_d & \mathbf{0} \\ x'_d & \mathbf{0} & y_d \\ \mathbf{0} & y'_d & z_d \end{pmatrix}, \end{aligned} \quad (4.54)$$

where $\arg(x'_q) = \arg(x_q^*)$ and $\arg(y'_q) = \arg(y_q^*)$ for $q = u$ and d . The texture of M_u and M_d can be regarded as a non-Hermitian extension of the six-texture-zero quark mass matrices prescribed first in Ref. [4]. To reproduce the measured magnitude of the flavor mixing matrix element V_{cb} (or V_{ts}), the restriction $|y_q| < |y'_q|$ is generally required. If the exotic hierarchy $|y_q| \ll |y'_q| \sim |z_q|$ is assumed, one obtains $|V_{cb}| \approx |V_{ts}| \sim \mathcal{O}(m_s/m_b)$ [66], which is consistent with the experimental measurement. On the other hand, the Cabibbo mixing angle (i.e., $|V_{us}|$ and $|V_{cd}|$) can properly be reproduced from (4.54) even in the assumption of $|x'_q| = |x_q|$.

As the mass matrices M_u and M_d consist totally of twelve independent parameters (two of them are the phase differences between M_u and M_d), it is not difficult to find out the parameter space allowed by current data of flavor mixing and CP violation. The number of free parameters could further be reduced to ten, e.g., by taking $\arg(y_u) = \arg(y_d)$ and $x'_q = x_q$ (for $q = u$ and d).

Triangular mass matrices It has been shown in Ref. [67] that the arbitrary up- and down-type mass matrices can always be transformed into a triangular form by proper rotations of the right-handed quark fields. If one of the two triangular mass matrices is diagonalized, the other can again be cast into the triangular form without loss of any generality. There totally exist four types of triangular textures with three zeros in the upper-left, upper-right, lower-left and lower-right corners of the mass matrix, depending on the chosen basis of flavor space. While the first two types do not yield simple relations between quark masses and flavor mixing angles, the latter two do [67, 68]. For example one may take

$$M_q = \begin{pmatrix} C_q & C'_q & C''_q \\ \mathbf{0} & B_q & B'_q \\ \mathbf{0} & \mathbf{0} & A_q \end{pmatrix}, \quad (4.55)$$

where $q = u$ and d , and all non-zero elements are generally complex.

One may diagonalize M_q with the help of the bi-unitary transformations, as shown in (3.1), but the flavor mixing matrix V depends only upon the unitary matrices used to rotate the left-handed quark fields. In the flavor basis where the up-type mass matrix is diagonal (i.e., $M_u = \text{Diag}\{m_u, m_c, m_t\}$), the triangular down-type mass matrix M_d can simply be expressed, in terms of the quark masses (m_d, m_s, m_b) and the matrix elements

of V , as follows:

$$M_d \approx \begin{pmatrix} m_d V_{ud} & m_s V_{us} & m_b V_{ub} \\ \mathbf{0} & m_s V_{cs} & m_b V_{cb} \\ \mathbf{0} & \mathbf{0} & m_b V_{tb} \end{pmatrix}. \quad (4.56)$$

This interesting pattern, valid up to the accuracy of $\mathcal{O}(\lambda^4)$ for $\lambda = \sin \theta_C \approx 0.22$ [68], reflects the hierarchy of quark masses and flavor mixing angles vividly. In the same way and to the same accuracy, we may choose the flavor basis in which M_d is diagonal and M_u takes the following triangular form:

$$M_u \approx \begin{pmatrix} m_u V_{ud}^* & m_c V_{cd}^* & m_t V_{td}^* \\ \mathbf{0} & m_c V_{cs}^* & m_t V_{ts}^* \\ \mathbf{0} & \mathbf{0} & m_t V_{tb}^* \end{pmatrix}. \quad (4.57)$$

As $m_d/m_s \sim m_s/m_b \sim \lambda^2$ and $m_u/m_c \sim m_c/m_t \sim \lambda^4$ hold, the approximate relations between flavor mixing angles and quark mass ratios can straightforwardly be obtained from diagonalizing either M_d in (4.56) or M_u in (4.57).

Pure phase mass matrices The pure phase pattern of quark mass matrices, given by

$$M_q = \frac{\chi_q}{3} \begin{pmatrix} e^{i\phi_{11}^q} & e^{i\phi_{12}^q} & e^{i\phi_{13}^q} \\ e^{i\phi_{21}^q} & e^{i\phi_{22}^q} & e^{i\phi_{23}^q} \\ e^{i\phi_{31}^q} & e^{i\phi_{32}^q} & e^{i\phi_{33}^q} \end{pmatrix} \quad (4.58)$$

with $q = u$ and d , comes from the phenomenological conjecture that all the Yukawa couplings of quarks have the identical moduli but their phases may be different [69]. This idea, usually referred to as the hypothesis of the universal strength for Yukawa couplings, is a complex variation of the democratic form of quark mass matrices in (4.44). Again the freedom of right-handed quark fields can be used to recast the pure phase mass matrices. One has verified that the Hermiticity of M_u and M_d in (4.58) must be abandoned, in order to accommodate the observed values of quark masses and flavor mixing parameters.

Further assumptions need be made for the generic pattern of pure phase mass matrices, otherwise there is little calculability to reproduce the measured spectrum of quark masses and flavor mixing angles. An interesting ansatz is to assume M_u and M_d to be symmetric and have the following parallel structures [70]:

$$\begin{aligned} M_u &= \frac{\chi_u}{3} \begin{pmatrix} 1 & 1 & e^{i\sigma_u} \\ 1 & 1 & e^{i\rho_u} \\ e^{i\sigma_u} & e^{i\rho_u} & e^{i\rho_u} \end{pmatrix}, \\ M_d &= \frac{\chi_d}{3} \begin{pmatrix} 1 & 1 & e^{i\sigma_d} \\ 1 & 1 & e^{i\rho_d} \\ e^{i\sigma_d} & e^{i\rho_d} & e^{i\rho_d} \end{pmatrix}. \end{aligned} \quad (4.59)$$

The six free parameters in M_u and M_d can be determined by six known quark masses. Therefore the three flavor mixing angles can be derived from (4.59) in terms of only four quark mass ratios (i.e., without additional free parameters). The problem associated with this ansatz or its variations is that the measured strength of CP violation in quark mixing cannot correctly be reproduced. To remedy this defect one has to slightly relax the universality constraint on the moduli of M_u or M_d , or both of them.

4.7 Mass matrices at superhigh energy scales

The phenomenological schemes of quark mass matrices can be studied not only at the experimentally accessible scales (e.g., $\mu \leq 10^2$ GeV) but also at the scales of string or grand unified theories (e.g., $\mu \geq 10^{15}$ GeV). For the latter case the running of quark masses and flavor mixing parameters from the superhigh scale to the weak-interaction scale must be investigated with the help of renormalization-group equations. It is in general expected that a particular scheme of mass matrices at one scale may be significantly changed at another scale, e.g., the original texture zeros become nonvanishing as a straightforward consequence of the running effect [64, 71]. In this section we first point out some generic running behaviors of quark mass ratios and flavor mixing angles. Then we take into account a specific ansatz of fermion mass matrices in the context of supersymmetric grand unified theories for illustration.

The one-loop evolution equations for the Yukawa coupling matrices of quarks and charged leptons, in the assumption of no threshold effect at the intermediate or superhigh scales, have been presented in Ref. [72]. They can considerably be simplified if the hierarchy of fermion masses and that of the CKM matrix elements are taken into account. In the leading order approximation one finds that the running behaviors of quark mass ratios and flavor mixing angles are governed only by the third-family Yukawa coupling eigenvalues f_t , f_b and f_τ . From a superhigh scale $\mu = M_X$ to the weak scale $\mu = M_Z$, the relevant evolution functions can be defined as

$$\xi_i = \exp \left[-\frac{1}{16\pi^2} \int_0^{\ln(M_X/M_Z)} f_i^2(\chi) d\chi \right] \quad (4.60)$$

with $i = t, b$ or τ and $\chi = \ln(\mu/M_Z)$, whose values depend on the specific model of spontaneous symmetry breaking below the scale M_X . We then arrive at the following results, which are valid to a good degree of accuracy:

$$\begin{aligned} \left[\frac{m_u}{m_c} \right]_{M_Z} &= \left[\frac{m_u}{m_c} \right]_{M_X} , \\ \left[\frac{m_d}{m_s} \right]_{M_Z} &= \left[\frac{m_d}{m_s} \right]_{M_X} , \\ \left[\frac{m_c}{m_t} \right]_{M_Z} &= \left[\frac{m_c}{m_t} \right]_{M_X} (\xi_t^x \xi_b^y) , \\ \left[\frac{m_s}{m_b} \right]_{M_Z} &= \left[\frac{m_s}{m_b} \right]_{M_X} (\xi_t^y \xi_b^x) ; \end{aligned} \quad (4.61)$$

and

$$\begin{aligned} [\tan \theta_u]_{M_Z} &= [\tan \theta_u]_{M_X} , \\ [\tan \theta_d]_{M_Z} &= [\tan \theta_d]_{M_X} , \\ [\sin \theta]_{M_Z} &= [\sin \theta]_{M_X} (\xi_t \xi_b)^y , \\ [\mathcal{J}]_{M_Z} &= [\mathcal{J}]_{M_X} (\xi_t \xi_b)^{2y} , \end{aligned} \quad (4.62)$$

where the coefficients $(x, y) = (-1.5, 1.5)$, $(-1.5, -0.5)$ and $(-3, -1)$ in the standard model, the two-Higgs doublet model and the minimal supersymmetric model, respectively.

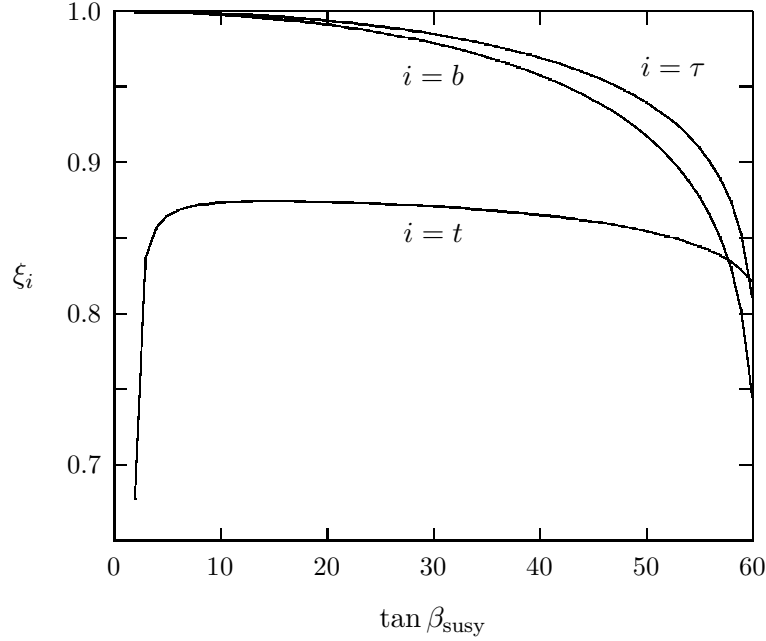


Figure 4.3: The magnitudes of ξ_t , ξ_b and ξ_τ changing with $\tan \beta_{\text{susy}}$ in the minimal supersymmetric standard model.

Some remarks are in order: (a) the evolution of m_u/m_c (or m_d/m_s) is negligibly small, thus m_u/m_t and m_c/m_t (or m_d/m_b and m_s/m_b) have the identical running effects; (b) among four parameters of the CKM matrix V (i.e., θ_u , θ_d , θ and φ), only the mixing angle θ may have significant evolution; (c) the running of the CP -violating parameter \mathcal{J} depends on that of $\sin^2 \theta$, but the three angles of the unitarity triangle Δ_s (i.e., α , β and γ) are essentially scale-independent.

For illustration we evaluate the running functions ξ_t , ξ_b and ξ_τ in the framework of the minimal supersymmetric standard model, which is best motivated by the present achievement towards a grand unified theory of all interactions. With the typical inputs $m_t(M_Z) = 175$ GeV, $m_b(M_Z) = 2.9$ GeV and $m_\tau(M_Z) = 1.777$ GeV, we calculate ξ_t , ξ_b and ξ_τ for arbitrary $\tan \beta_{\text{susy}}$ (the ratio of Higgs vacuum expectation values) from $M_X = 10^{16}$ GeV to $M_Z = 91.187$ GeV. The result is shown in Fig. 4.3. One can see that $\xi_b \approx \xi_\tau \approx 1$ for $\tan \beta_{\text{susy}} \leq 20$. Within the perturbatively allowed region ξ_b and ξ_τ may be comparable in magnitude with ξ_t when $\tan \beta_{\text{susy}} \geq 50$. In this case the evolutions of quark masses and flavor mixing angles are sensitive to both t - and b -quark contributions, and to the τ -lepton contribution in a hidden way.

Let us define the running factors of m_c/m_t , m_s/m_b , $\sin \theta$ and \mathcal{J} in (4.61) and (4.62) as R_1 , R_2 , R_3 and R_4 respectively. Their magnitudes can then be obtained from the result of ξ_t and ξ_b in the minimal supersymmetric standard model (with $x = -3$ and $y = -1$), as shown in Fig. 4.4. We observe that these evolution factors are quite stable for a large range of the ratio of Higgs vacuum expectation values, i.e., $5 \leq \tan \beta_{\text{susy}} \leq 50$. Such features are essentially independent of the specific structures of quark mass matrices.

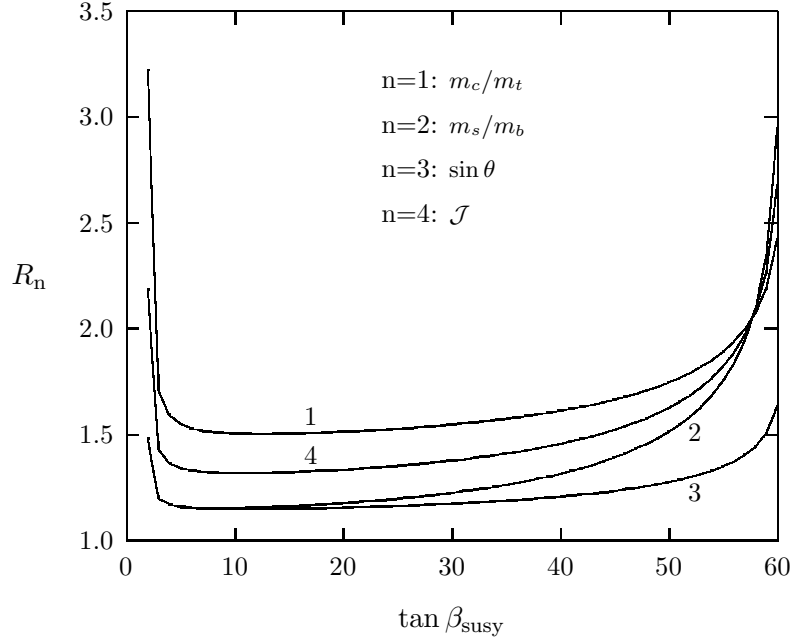


Figure 4.4: The running factors R_n (from M_X to M_Z) changing with $\tan \beta_{\text{susy}}$ in the minimal supersymmetric standard model.

Now we concentrate on specific patterns of quark and charged lepton mass matrices at the scale of supersymmetric grand unified theories ($M_X = 10^{16}$ GeV). In view of the success of the Hermitian quark mass matrices with four texture zeros in accounting for the low-energy flavor mixing phenomena, we take the following ansatz at the scale M_X :

$$\begin{aligned}
M_u &= \begin{pmatrix} \mathbf{0} & +ix & \mathbf{0} \\ -ix & y & ry \\ \mathbf{0} & ry & z \end{pmatrix}, \\
M_d &= \begin{pmatrix} \mathbf{0} & x' & \mathbf{0} \\ x' & y' & ry' \\ \mathbf{0} & ry' & z' \end{pmatrix}, \\
M_e &= \begin{pmatrix} \mathbf{0} & x' & \mathbf{0} \\ x' & -3y' & ry' \\ \mathbf{0} & ry' & z' \end{pmatrix},
\end{aligned} \tag{4.63}$$

where $|x| \ll |y| \ll |z|$, $|x'| \ll |y'| \ll |z'|$, and r is a constant of $\mathcal{O}(1)$. The texture zeros of $M_{u,d,e}$ as well as the relationship between M_d and M_e can naturally be obtained in a variety of grand unified models where the down quarks and the charged leptons lie in the same multiplet [73]. For example, the coupling of a Higgs boson in the **10** plets of an $\text{SO}(10)$ model gives certain entries in the Yukawa coupling matrices of the form $(M_d)_{ij} = (M_e)_{ij}$ (for $i \neq j$), while a Higgs boson in the **126** plets yields $(M_d)_{22} = -3(M_e)_{22}$. For simplicity we have taken the phase difference between $(M_u)_{12}$ and $(M_d)_{12}$ to be $\pi/2$, a value favored by current data on CP violation. The constant r may take values such as 1, $\sqrt{2}$ or 2, from

the phenomenological point of view. It is obvious that the ansatz (4.63) totally involves six free parameters (x, y, z and x', y', z'), which can be determined from the inputs of three up-type quark masses and three charged lepton masses. Thus it is able to give seven predictions at the weak scale M_Z , three for the down-type quark masses and four for the parameters of flavor mixing and CP violation. Of course some of these predictions depend sensitively upon the unknown value of $\tan \beta_{\text{susy}}$.

The predictions of this ansatz for the down-type quark masses read, at the scale M_X and in the next-to-leading order approximation, as follows:

$$\begin{aligned} m_d &= 3m_e \left(1 + \frac{4r^2}{9} \cdot \frac{m_\mu}{m_\tau} \right), \\ m_s &= \frac{m_\mu}{3} \left(1 - \frac{4r^2}{9} \cdot \frac{m_\mu}{m_\tau} \right), \\ m_b &= m_\tau. \end{aligned} \tag{4.64}$$

To renormalize these results down to the weak scale M_Z , we need to take into account the evolutions of charged lepton masses

$$\begin{aligned} \left[\frac{m_e}{m_\mu} \right]_{M_Z} &= \left[\frac{m_e}{m_\mu} \right]_{M_X}, \\ \left[\frac{m_\mu}{m_\tau} \right]_{M_Z} &= \left[\frac{m_\mu}{m_\tau} \right]_{M_X} (\xi_\tau^{-3}) \end{aligned} \tag{4.65}$$

and make use of the following running function in the framework of the minimal supersymmetric standard model:

$$\zeta_{\text{de}} = \exp \left[+ \frac{1}{16\pi^2} \int_0^{\ln(M_X/M_Z)} G_{\text{de}}(\chi) d\chi \right], \tag{4.66}$$

where $G_{\text{de}} = G_d - G_e$ with G_d and G_e given in terms of the gauge couplings g_1, g_2 and g_3 in Ref. [64]. With the inputs $g_1^2 = 0.127$, $g_2^2 = 0.42$ and $g_3^2 = 1.44$ at the scale M_Z , one finds $\zeta_{\text{de}} = 2.27$. The down-type quark masses at the weak scale turn out to be

$$\begin{aligned} m_d &= 3m_e \left(1 + \frac{4r^2}{9} \cdot \frac{m_\mu}{m_\tau} \xi_\tau^3 \right) \zeta_{\text{de}}, \\ m_s &= \frac{m_\mu}{3} \left(1 - \frac{4r^2}{9} \cdot \frac{m_\mu}{m_\tau} \xi_\tau^3 \right) \zeta_{\text{de}}, \\ m_b &= m_\tau \frac{\xi_t \xi_b^3}{\xi_\tau^3} \zeta_{\text{de}}. \end{aligned} \tag{4.67}$$

Taking $r^2 = 2$ and $\tan \beta_{\text{susy}} = 50$ for example, we obtain $m_d \approx 3.6$ MeV, $m_s \approx 76$ MeV and $m_b \approx 3.2$ GeV, essentially in agreement with the results listed in (2.9).

The predictions of the ansatz (4.63) for flavor mixing and CP violation at the weak scale M_Z can be obtained with the help of the renormalization relations in (4.61) and

(4.62). To the leading order accuracy we arrive at

$$\begin{aligned}\tan \theta_u &= \sqrt{\frac{m_u}{m_c}}, \\ \tan \theta_d &= \sqrt{\frac{m_d}{m_s}};\end{aligned}\tag{4.68}$$

and

$$\begin{aligned}\sin \theta &= |r| \left(\frac{m_s}{m_b} \xi_b^2 - \frac{m_c}{m_t} \xi_t^2 \right), \\ \mathcal{J} &= r^2 \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{m_d}{m_s}} \left(\frac{m_s}{m_b} \right)^2 \xi_b^4.\end{aligned}\tag{4.69}$$

One can see again that the mixing angles θ_u and θ_d are basically scale-independent. The running effects of θ and \mathcal{J} depend mainly upon the change of ξ_b from M_X to M_Z , which becomes significant only for $\tan \beta_{\text{susy}} \geq 30$. On this point we conclude that the ansatz (4.63) for fermion mass matrices has instructive predictions and is favored by current data. It might be a natural consequence of a complete grand unified model involving some family symmetries at superhigh scales.

A few similar schemes of fermion mass matrices, based also on the grand unification ideas, have been proposed in the literature (see, e.g., Refs. [74, 75]). For a brief review of these works, we refer the reader to Ref. [48].

5 Lepton mass matrices and neutrino oscillations

5.1 Classification of 3×3 neutrino mass matrices

Now we turn to the phenomenological schemes of lepton mass matrices M_l and M_ν , which can naturally lead to hierarchical neutrino mass-squared differences and large lepton flavor mixing angles as required by current neutrino oscillation data. Attempts in this direction are of course restricted by both the preliminary experimental knowledge and the premature theoretical insight that one has today for neutrino physics. Hence we shall pay particular attention to the possible underlying symmetries of lepton flavors, from which realistic models of lepton mass generation and flavor mixing could simply be constructed. The possibility of leptonic CP violation, which is a crucial ingredient of leptogenesis to explain the cosmological baryon asymmetry [76], will also be discussed.

Before introducing some phenomenologically favored patterns of lepton mass matrices, we first give a rough but instructive classification of 3×3 neutrino mass matrices in the flavor basis where the charged lepton mass matrix is diagonal. To do so the LSND evidence for $\nu_\mu \rightarrow \nu_e$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ oscillations is tentatively put aside.

As briefly summarized in section 2.2, the observed neutrino anomalies can most naturally be interpreted by the hypothesis of neutrino oscillations, indicating that neutrinos are massive and lepton flavors are mixed. In the framework of three light neutrinos, to interpret current atmospheric and solar neutrino oscillation data requires $\Delta m_{21}^2 = \Delta m_{\text{sun}}^2 \ll$

$\Delta m_{\text{atm}}^2 = \Delta m_{32}^2$ and $|V_{e3}|^2 \ll 1$; i.e., the solar and atmospheric neutrino oscillations are respectively dominated by $\nu_\mu \rightarrow \nu_e$ and $\nu_\mu \rightarrow \nu_\tau$ transitions, and thus approximately decoupled. Given the large hierarchy between Δm_{21}^2 and Δm_{32}^2 , there are three different possibilities for the spectrum of neutrino masses [77]:

$$\begin{aligned} \text{(a)} : \quad & m_1, m_2 \ll m_3 ; \\ \text{(b)} : \quad & m_1 \approx m_2 \gg m_3 ; \\ \text{(c)} : \quad & m_1 \approx m_2 \approx m_3 . \end{aligned} \tag{5.1}$$

In case (a) the relative magnitude of m_1 and m_2 is not restricted; in case (b) we require the inequality $|m_2 - m_1| \ll m_{1,2}$; and in case (c) the inequality $|m_2 - m_1| \ll |m_3 - m_2|$ should be satisfied. Neglecting possible CP -violating phases and taking $|V_{e3}| = 0$ in the leading order approximation, one can parametrize the lepton flavor mixing in terms of two rotation angles:

$$V \approx \begin{pmatrix} c_\odot & -s_\odot & \mathbf{0} \\ s_\odot c_\bullet & c_\odot c_\bullet & -s_\bullet \\ s_\odot s_\bullet & c_\odot s_\bullet & c_\bullet \end{pmatrix}, \tag{5.2}$$

where $s_\odot \equiv \sin \theta_{\text{sun}}$, $c_\bullet \equiv \cos \theta_{\text{atm}}$, and so on. Obviously θ_{sun} and θ_{atm} correspond to the mixing angles of solar and atmospheric neutrino oscillations. The present experimental data favor $\sin^2 2\theta_{\text{atm}} > 0.8$ or $\theta_{\text{atm}} \sim 32^\circ - 45^\circ$; while $\sin^2 2\theta_{\text{sun}}$ may be either small ($\theta_{\text{sun}} \sim 1^\circ - 3^\circ$, small-angle MSW solution) or large ($\theta_{\text{sun}} \sim 27^\circ - 45^\circ$, large-angle MSW solution), or nearly maximal ($\theta_{\text{sun}} \sim 45^\circ$, vacuum oscillation).

In order to build realistic models of lepton mass generation which lead to the flavor mixing pattern (5.2), it is phenomenologically useful to figure out possible forms of the neutrino mass matrix that are compatible with the mass spectra in (5.1). One should keep in mind that the flavor mixing matrix V arises from the mismatch between the diagonalization of the charged lepton mass matrix M_l and that of the neutrino mass matrix M_ν in a specific flavor basis. Hence the flavor mixing angles θ_{sun} and θ_{atm} depend in general on both the neutrino masses and the charged lepton masses. Even though the latter exhibit a strong hierarchy in magnitude (see (2.1) for illustration), their contributions to θ_{sun} and (or) θ_{atm} may be non-negligible in some cases. For instance, the small MSW-type mixing angle θ_{sun} could be dominated by a contribution of order $\arcsin(\sqrt{m_e/m_\mu}) \approx 4^\circ$ from the charged lepton sector, if M_l takes the Hermitian texture with full nearest-neighbor mixing [4]. The large mixing angle θ_{atm} might get a significant contribution of order $\arcsin(\sqrt{m_\mu/m_\tau}) \approx 14^\circ$ (i.e., about 30% of $\theta_{\text{atm}} \sim 45^\circ$) in the same scenario of M_l . It is therefore improper, even misleading, to account for the lepton flavor mixing solely in terms of the mixing parameters in the neutrino sector.

The observation made above has an important implication: if one writes the neutrino mass matrix M_ν taking into account the flavor mixing matrix V in the basis where the charged lepton mass matrix M_l is diagonalized, the resultant M_ν remains dependent on M_l in a hidden way (i.e., through unspecified flavor mixing angles). In this sense the useful information that one can obtain about the textures of M_l is limited. Nevertheless it should still be instructive to find out the possible leading order forms of M_ν , versus the diagonal M_l , at the present phenomenological stage. Such forms of M_ν might provide useful hints for model buildings, which could finally shed light on the dynamics of lepton

Table 5.1: Leading order textures of the neutrino mass matrix M_ν , in which $s_\odot \equiv \sin \theta_{\text{sun}}$, $c_{2\bullet} \equiv \cos 2\theta_{\text{atm}}$, and so on; and “0” or “1” only means $\mathcal{O}(0)$ or $\mathcal{O}(1)$ in magnitude.

| Masses $\{\lambda_1, \lambda_2, \lambda_3\}$ | M_ν (small θ_{sun}) | M_ν (large θ_{sun}) |
|--|---|--|
| (a): $\{0, 0, 1\}m_3$ | $m_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_\bullet^2 & -s_\bullet c_\bullet \\ 0 & -s_\bullet c_\bullet & c_\bullet^2 \end{pmatrix}$ | $m_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_\bullet^2 & -s_\bullet c_\bullet \\ 0 & -s_\bullet c_\bullet & c_\bullet^2 \end{pmatrix}$ |
| (b): $\{1, -1, 0\}m_1$ | $m_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -c_\bullet^2 & -s_\bullet c_\bullet \\ 0 & -s_\bullet c_\bullet & -s_\bullet^2 \end{pmatrix}$ | $m_1 \begin{pmatrix} c_{2\odot} & s_{2\odot} c_\bullet & s_{2\odot} c_\bullet \\ s_{2\odot} c_\bullet & -c_{2\odot} c_\bullet^2 & -c_{2\odot} s_\bullet c_\bullet \\ s_{2\odot} c_\bullet & -c_{2\odot} s_\bullet c_\bullet & -c_{2\odot} s_\bullet^2 \end{pmatrix}$ |
| (b): $\{1, 1, 0\}m_1$ | $m_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\bullet^2 & s_\bullet c_\bullet \\ 0 & s_\bullet c_\bullet & s_\bullet^2 \end{pmatrix}$ | $m_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\bullet^2 & s_\bullet c_\bullet \\ 0 & s_\bullet c_\bullet & s_\bullet^2 \end{pmatrix}$ |
| (c): $\{1, 1, 1\}m_1$ | $m_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $m_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ |
| (c): $\{-1, 1, 1\}m_1$ | $m_1 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $m_1 \begin{pmatrix} -c_{2\odot} & -s_{2\odot} c_\bullet & -s_{2\odot} c_\bullet \\ -s_{2\odot} c_\bullet & s_\bullet^2 & -s_\bullet c_\bullet \\ -s_{2\odot} c_\bullet & -s_\bullet c_\bullet & c_\bullet^2 \end{pmatrix}$ |
| (c): $\{1, -1, 1\}m_1$ | $m_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -c_\bullet & -s_\bullet \\ 0 & -s_\bullet & c_\bullet \end{pmatrix}$ | $m_1 \begin{pmatrix} c_{2\odot} & s_{2\odot} c_\bullet & s_{2\odot} c_\bullet \\ s_{2\odot} c_\bullet & s_\bullet^2 & -s_\bullet c_\bullet \\ s_{2\odot} c_\bullet & -s_\bullet c_\bullet & c_\bullet^2 \end{pmatrix}$ |
| (c): $\{1, 1, -1\}m_1$ | $m_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\bullet & s_\bullet \\ 0 & s_\bullet & -c_\bullet \end{pmatrix}$ | $m_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\bullet & s_\bullet \\ 0 & s_\bullet & -c_\bullet \end{pmatrix}$ |

mass generation and flavor mixing. Therefore we proceed to make a classification of the textures of M_ν allowed by current neutrino oscillation data.

In doing so we do not consider the origin of M_ν , no matter how it could originate from the seesaw mechanism or from other flavor symmetries. But for simplicity we shall assume CP invariance in the lepton sector, thus M_ν is a real symmetric matrix for either Dirac- or Majorana-type neutrinos. In the flavor basis where M_l is diagonal, M_ν takes the following form

$$M_\nu = V^* \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} V^\dagger, \quad (5.3)$$

in which λ_i are the neutrino mass eigenvalues ($|\lambda_i| = m_i$), and V is the 3×3 flavor mixing matrix. Taking the neutrino mass spectra in (5.1) and the approximate flavor mixing pattern in (5.2) into account, we obtain fourteen different textures of M_ν for the case of small θ_{sun} (small-angle MSW solution) and that of large θ_{sun} (large-angle MSW solution or vacuum oscillation solution). The results are listed in Table 5.1 for illustration. Note

that all texture zeros in M_ν only imply small quantities, and all unity elements of M_ν only mean $\mathcal{O}(1)$ in magnitude. The matrix elements of $\mathcal{O}(0)$, in particular those in case (c) with $\lambda_1 \approx \lambda_2 \approx \lambda_3$, play an important role to generate large flavor mixing angles. If $\theta_{\text{atm}} = \theta_{\text{sun}} = 45^\circ$ (bi-maximal mixing pattern [78]) is taken, the elements of M_ν are proportional to algebraic numbers of $\mathcal{O}(1)$. The same situation appears for the special choices $\theta_{\text{atm}} = 45^\circ$ and $\theta_{\text{sun}} = 0^\circ$ [77].

In principle one could also classify the possible forms of the charged lepton mass matrix M_l in the flavor basis where the neutrino mass matrix M_ν is diagonal. Without much prejudice a more general but incomplete classification of the leading order textures of lepton mass matrices (both M_ν and M_l) have been presented in Ref. [79].

5.2 Lepton mixing from flavor democracy breaking

The idea of subnuclear flavor democracy and its explicit breaking has been applied to the quark sector (see section 4.5) to interpret the strong mass hierarchy of up- and down-type quarks. Since the mass spectrum of charged leptons exhibits a similar hierarchy as that of quarks, it would be natural to consider the same symmetry limit for the charged lepton mass matrix, i.e., $M_l^{(0)}$ takes the same form as $M_q^{(0)}$ in (4.44). As for the neutrino sector, we have no direct information about the absolute values or relative magnitudes of neutrino masses. The observed large (nearly maximal) mixing angles and the tiny mass-squared differences in atmospheric and solar neutrino oscillations, however, favor the possibility that the three neutrino masses are approximately degenerate, although the other possibilities for the neutrino mass spectrum cannot be excluded. We therefore start with the hypothesis of the flavor democracy for charged leptons and the mass degeneracy for neutrinos to construct a phenomenological model of lepton mass generation and flavor mixing [27]. CP violation can be incorporated into this model, when complex symmetry breaking terms are explicitly introduced.

In the limits of the flavor democracy and the mass degeneracy, the charged lepton and neutrino mass matrices can respectively be written as

$$\begin{aligned} M_l^{(0)} &= \frac{C_l}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \\ M_\nu^{(0)} &= C_\nu \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{5.4}$$

where $C_l = m_\tau$ and $C_\nu = m_0$ measure the corresponding mass scales. If the three neutrinos are of the Majorana type, $M_\nu^{(0)}$ could take a more general form $M_\nu^{(0)} P_\nu$ with $P_\nu = \text{Diag}\{1, e^{i\phi_1}, e^{i\phi_2}\}$. As the Majorana phase matrix P_ν has no effect on the flavor mixing and CP -violating observables in neutrino oscillations, it will be neglected in the subsequent discussions. The neutrino mass matrix $M_\nu^{(0)}$ exhibits an $S(3)$ symmetry, while the charged lepton mass matrix $M_l^{(0)}$ exhibits an $S(3)_L \times S(3)_R$ symmetry.

One can transform the charged lepton mass matrix from the democratic basis $M_l^{(0)}$ into the hierarchical basis M_{0l} given in (2.2) through a simple orthogonal transformation

$M_{0l} = U_0 M_l^{(0)} U_0^T$, where U_0 has been shown in (4.45). Clearly $m_e = m_\mu = 0$, as seen in M_{0l} ; and $m_1 = m_2 = m_3 = m_0$ in $M_\nu^{(0)}$. There is no flavor mixing in this symmetry limit.

Of course there are a number of different ways to break the flavor democracy of $M_l^{(0)}$ (see Table 4.3) and the mass degeneracy of $M_\nu^{(0)}$. Whether the resultant flavor mixing patterns are compatible with the neutrino oscillation data is, at present, the only phenomenological criterion to select the proper symmetry breaking scenarios. In the following we take three instructive examples, which can respectively lead to the nearly bi-maximal mixing pattern, the small-versus-large mixing pattern, and the bi-maximal mixing pattern for atmospheric and solar neutrino oscillations.

Nearly bi-maximal mixing pattern A real diagonal breaking of the $S(3)_L \times S(3)_R$ symmetry of $M_l^{(0)}$ and the $S(3)$ symmetry of $M_\nu^{(0)}$ can lead to a nearly bi-maximal flavor mixing pattern, as first discussed in Ref. [27]. To incorporate CP violation into this neutrino mixing scenario, however, complex perturbative terms are required for $M_l^{(0)}$ [39]. Let us proceed with two different symmetry-breaking steps.

(a) Small real perturbations to the (3,3) elements of $M_l^{(0)}$ and $M_\nu^{(0)}$ are respectively introduced:

$$\begin{aligned}\Delta M_l^{(1)} &= \frac{C_l}{3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon_l \end{pmatrix}, \\ \Delta M_\nu^{(1)} &= C_\nu \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon_\nu \end{pmatrix}.\end{aligned}\tag{5.5}$$

In this case the charged lepton mass matrix $M_l^{(1)} = M_l^{(0)} + \Delta M_l^{(1)}$ remains symmetric under an $S(2)_L \times S(2)_R$ transformation, and the neutrino mass matrix $M_\nu^{(1)} = M_\nu^{(0)} + \Delta M_\nu^{(1)}$ has an $S(2)$ symmetry. The muon becomes massive (i.e., $m_\mu \approx 2|\varepsilon_l|m_\tau/9$), and the mass eigenvalue m_3 is no more degenerate with m_1 and m_2 (i.e., $|m_3 - m_0| = m_0|\varepsilon_\nu|$). After the diagonalization of $M_l^{(1)}$ and $M_\nu^{(1)}$, one finds that the 2nd and 3rd lepton families have a definite flavor mixing angle θ . We obtain $\tan \theta \approx -\sqrt{2}$ if the small correction of $\mathcal{O}(m_\mu/m_\tau)$ is neglected. Then neutrino oscillations at the atmospheric scale may arise in $\nu_\mu \rightarrow \nu_\tau$ transitions with $\Delta m_{32}^2 = \Delta m_{31}^2 \approx 2m_0|\varepsilon_\nu|$. The corresponding mixing factor $\sin^2 2\theta \approx 8/9$ is in good agreement with current data.

The symmetry breaking term $\Delta M_l^{(1)}$ given in (5.5) for the charged lepton mass matrix serves as a good illustrative example. One could consider a more general perturbation, analogous to $\Delta M_q^{(1)}$ given in (4.47) for quarks, in order to break the $S(3)_L \times S(3)_R$ symmetry of $M_l^{(0)}$ to an $S(2)_L \times S(2)_R$ symmetry. In this case we keep the perturbation to $M_\nu^{(0)}$ to be the same as $\Delta M_\nu^{(1)}$ in (5.5). Then it is easy to check that the leading order result for lepton flavor mixing (i.e., $\tan \theta \approx -\sqrt{2}$ or $\sin^2 2\theta \approx 8/9$) remains unchanged.

(b) Small imaginary perturbations, which have the identical magnitude but the opposite signs, are introduced to the (2,2) and (1,1) elements of $M_l^{(1)}$. For $M_\nu^{(1)}$ the corre-

sponding real perturbations are taken into account [39]:

$$\begin{aligned}\Delta M_l^{(2)} &= \frac{C_l}{3} \begin{pmatrix} -i\delta_l & 0 & 0 \\ 0 & i\delta_l & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \Delta M_\nu^{(2)} &= C_\nu \begin{pmatrix} -\delta_\nu & 0 & 0 \\ 0 & \delta_\nu & 0 \\ 0 & 0 & 0 \end{pmatrix}.\end{aligned}\tag{5.6}$$

We obtain $m_e \approx |\delta_l|^2 m_\tau^2 / (27 m_\mu)$ and $m_{1,2} = m_0(1 \mp \delta_\nu)$. The diagonalization of $M_l^{(2)} = M_l^{(1)} + \Delta M_l^{(2)}$ and $M_\nu^{(2)} = M_\nu^{(1)} + \Delta M_\nu^{(2)}$ leads to a full 3×3 flavor mixing matrix, which links neutrino mass eigenstates (ν_1, ν_2, ν_3) to neutrino flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$ in the following manner:

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} + i \xi_V \sqrt{\frac{m_e}{m_\mu}} + \zeta_V \frac{m_\mu}{m_\tau},\tag{5.7}$$

where

$$\begin{aligned}\xi_V &= \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \zeta_V &= \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{12}} & \frac{-1}{\sqrt{12}} & \frac{1}{\sqrt{3}} \end{pmatrix}.\end{aligned}\tag{5.8}$$

Note that the leading term of V is just the orthogonal matrix U_0 in (4.45), used to transform a democratic mass matrix into its counterpart in the hierarchical basis.

The diagonal non-Hermitian perturbation $\Delta M_l^{(2)}$ in (5.6) is certainly not the only way to generate large flavor mixing and CP violation of the pattern (5.7). One can find other two off-diagonal non-Hermitian perturbations as well as three Hermitian perturbations, as listed in Table 5.2, which are able to give rise to the same flavor mixing pattern V in the next-to-leading order approximation. All these six perturbations have a common feature: the (1,1) elements of their counterparts in the hierarchical basis all vanish [39]. This feature assures that the CP -violating effects, resulted from the complex perturbations in Table 5.2, are approximately independent of other details of the flavor symmetry breaking and have the identical strength to a high degree of accuracy. Hence it is in practice difficult to distinguish one scenario from another. The simplicity of the diagonal non-Hermitian perturbation $\Delta M_l^{(2)}$ and its parallelism with $\Delta M_\nu^{(2)}$ seem quite instructive for building a possible model of lepton mass generation. The Hermitian forms of $\Delta M_l^{(2)}$, on the other hand, have much similarity with the favored perturbations generating CP violation in the quark sector and could provide us some useful hints towards an underlying symmetry between the quarks and the charged leptons.

Table 5.2: Six imaginary perturbations $\Delta M_l^{(2)}$ for the charged lepton mass matrix.

| Hermitian form of $\Delta M_l^{(2)}$ | Non-Hermitian form of $\Delta M_l^{(2)}$ |
|--|--|
| $\frac{C_l}{3} \begin{pmatrix} 0 & -i\delta_l & 0 \\ i\delta_l & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\frac{C_l}{3} \begin{pmatrix} -i\delta_l & 0 & 0 \\ 0 & i\delta_l & 0 \\ 0 & 0 & 0 \end{pmatrix}$ |
| $\frac{C_l}{3} \begin{pmatrix} 0 & 0 & i\delta_l \\ 0 & 0 & -i\delta_l \\ -i\delta_l & i\delta_l & 0 \end{pmatrix}$ | $\frac{C_l}{3} \begin{pmatrix} 0 & 0 & i\delta_l \\ 0 & 0 & -i\delta_l \\ i\delta_l & -i\delta_l & 0 \end{pmatrix}$ |
| $\frac{C_l}{3} \begin{pmatrix} 0 & -i\delta_l & i\delta_l \\ i\delta_l & 0 & -i\delta_l \\ -i\delta_l & i\delta_l & 0 \end{pmatrix}$ | $\frac{C_l}{3} \begin{pmatrix} -i\delta_l & 0 & i\delta_l \\ 0 & i\delta_l & -i\delta_l \\ i\delta_l & -i\delta_l & 0 \end{pmatrix}$ |

The flavor mixing matrix V can in general be parametrized in terms of three Euler angles and one CP -violating phase⁶. A suitable parametrization, analogous to that in (3.29) for quark mixing, reads as follows:

$$\begin{aligned}
V &= \begin{pmatrix} c_l & s_l & 0 \\ -s_l & c_l & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\phi} & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} \begin{pmatrix} c_\nu & -s_\nu & 0 \\ s_\nu & c_\nu & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} s_l s_\nu c + c_l c_\nu e^{-i\phi} & s_l c_\nu c - c_l s_\nu e^{-i\phi} & s_l s \\ c_l s_\nu c - s_l c_\nu e^{-i\phi} & c_l c_\nu c + s_l s_\nu e^{-i\phi} & c_l s \\ -s_\nu s & -c_\nu s & c \end{pmatrix}, \tag{5.9}
\end{aligned}$$

in which $s_l \equiv \sin \theta_l$, $s_\nu \equiv \sin \theta_\nu$, $c \equiv \cos \theta$, and so on. The three mixing angles can all be arranged to lie in the first quadrant, while the CP -violating phase may take values between 0 and 2π . With the inputs $m_e/m_\mu \approx 0.00484$ and $m_\mu/m_\tau \approx 0.0594$ [1] we numerically obtain

$$\theta_l \approx 4^\circ, \quad \theta_\nu \approx 45^\circ, \quad \theta \approx 52^\circ, \quad \phi \approx 90^\circ. \tag{5.10}$$

The smallness of θ_l is a natural consequence of the mass hierarchy in the charged lepton sector, just as the smallness of θ_u in quark mixing [8]. On the other hand, both θ_ν and θ are too large to be comparable with the corresponding quark mixing angles (i.e., θ_d and θ as defined in section 3.6), reflecting the qualitative difference between quark and lepton flavor mixing phenomena. It is worth emphasizing that the leptonic CP -violating phase ϕ takes a special value ($\approx 90^\circ$) in our model. The same possibility is also favored for the quark mixing phenomenon in a variety of realistic mass matrices (see Refs. [8, 47] and section 4.3). Therefore maximal leptonic CP violation, in the sense that the magnitude of $\mathcal{J}_l = s_l c_l s_\nu c_\nu s^2 c \sin \phi$ is maximal for the fixed values of three flavor mixing angles, appears naturally as in the quark sector.

⁶For neutrinos of the Majorana type, two additional CP -violating phases may enter. But they are irrelevant to neutrino oscillations and can be ignored for our present purpose.

Some consequences of this lepton mixing scenario are as follows:

(1) The mixing pattern in (5.7), after neglecting small corrections from the charged lepton masses, is quite similar to that of the pseudoscalar mesons π^0 , η and η' in QCD in the limit of the chiral $SU(3)_L \times SU(3)_R$ symmetry:

$$\begin{aligned} |\pi^0\rangle &= \frac{1}{\sqrt{2}} (|\bar{u}u\rangle - |\bar{d}d\rangle) , \\ |\eta\rangle &= \frac{1}{\sqrt{6}} (|\bar{u}u\rangle + |\bar{d}d\rangle - 2|\bar{s}s\rangle) , \\ |\eta'\rangle &= \frac{1}{\sqrt{3}} (|\bar{u}u\rangle + |\bar{d}d\rangle + |\bar{s}s\rangle) . \end{aligned} \quad (5.11)$$

A theoretical derivation of the flavor mixing matrix $V \approx U_0$ has been given in Ref. [80], in the framework of a left-right symmetric extension of the standard model with $S(3)$ and $Z(4) \times Z(3) \times Z(2)$ symmetries.

(2) The V_{e3} element, of magnitude

$$|V_{e3}| = \frac{2}{\sqrt{6}} \sqrt{\frac{m_e}{m_\mu}} , \quad (5.12)$$

is naturally suppressed in the symmetry breaking scheme outlined above. A similar feature appears in the 3×3 quark flavor mixing matrix, i.e., $|V_{ub}|$ is the smallest among the nine quark mixing elements. Indeed the smallness of V_{e3} provides a necessary condition for the decoupling of solar and atmospheric neutrino oscillations, even though neutrino masses are nearly degenerate. The effect of small but nonvanishing V_{e3} will manifest itself in long-baseline $\nu_\mu \rightarrow \nu_e$ and $\nu_e \rightarrow \nu_\tau$ transitions, as shown in Ref. [28].

(3) The flavor mixing between the 1st and 2nd lepton families and that between the 2nd and 3rd lepton families are nearly maximal. This property, together with the natural smallness of $|V_{e3}|$, allows a satisfactory interpretation of the observed large mixing in atmospheric and solar neutrino oscillations. One obtains ⁷

$$\begin{aligned} \sin^2 2\theta_{\text{sun}} &= 1 - \frac{4}{3} \frac{m_e}{m_\mu} , \\ \sin^2 2\theta_{\text{atm}} &= \frac{8}{9} \left(1 + \frac{m_\mu}{m_\tau} \right) \end{aligned} \quad (5.13)$$

to a high degree of accuracy; i.e., $\sin^2 2\theta_{\text{sun}} \approx 0.99$ and $\sin^2 2\theta_{\text{atm}} \approx 0.94$, in agreement with current data [18]. It is obvious that the model is fully consistent with the vacuum oscillation solution to the solar neutrino problem [81] and might also be able to incorporate the large-angle MSW solution [82].

(4) The rephasing-invariant strength of CP violation in (3.14), defined in close analogy with that for quarks, is given as

$$\mathcal{J}_l = \frac{1}{3\sqrt{3}} \sqrt{\frac{m_e}{m_\mu}} \left(1 + \frac{1}{2} \frac{m_\mu}{m_\tau} \right) . \quad (5.14)$$

⁷In calculating $\sin^2 2\theta_{\text{sun}}$ we have taken into account the $\mathcal{O}(m_e/m_\mu)$ correction to the expression of the lepton flavor mixing matrix V in (5.7).

Explicitly we have $\mathcal{J}_l \approx 1.4\%$. The large magnitude of \mathcal{J}_l for lepton mixing is very non-trivial, as the same quantity for quark mixing is only of order 10^{-5} (see Refs. [8, 47] and section 4.3). If the mixing pattern under discussion is reconciled with the large-angle MSW solution to the solar neutrino problem, then the relevant CP - or T -violating signals should be large enough to be measured from the asymmetry between $P(\nu_\mu \rightarrow \nu_e)$ and $P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e)$ or that between $P(\nu_\mu \rightarrow \nu_e)$ and $P(\nu_e \rightarrow \nu_\mu)$ in the long-baseline neutrino experiments (see section 5.6 for detailed discussions).

(5) Finally it is worth remarking that our lepton mixing pattern is not necessarily in conflict with current constraints on the neutrinoless double beta decay [17, 83], if neutrinos are of the Majorana type. In the presence of CP violation, the effective mass term of the $(\beta\beta)_{0\nu}$ decay can simply be written as $\langle m_\nu \rangle = \sum (m_i \hat{V}_{ei}^2)$, where $\hat{V} = VP_\nu$ and $P_\nu = \text{Diag}\{1, e^{i\phi_1}, e^{i\phi_2}\}$ is the Majorana phase matrix. If the unknown phases are taken to be $\phi_1 = \phi_2 = 90^\circ$, for example, one arrives at

$$|\langle m_\nu \rangle| = \frac{2}{\sqrt{3}} \sqrt{\frac{m_e}{m_\mu}} m_i, \quad (5.15)$$

in which $m_i \sim 1 - 2$ eV (for $i = 1, 2, 3$) as required by the near degeneracy of three neutrinos in our model to accommodate the hot dark matter of the universe. Obviously $|\langle m_\nu \rangle| \approx 0.08 m_i \leq 0.2$ eV, the latest bound of the $(\beta\beta)_{0\nu}$ decay given in (2.16).

Small-versus-large mixing pattern To generate a small mixing angle for the solar neutrino oscillation and a large one for the atmospheric neutrino oscillation, we turn to a different symmetry-breaking scenario for the charged lepton mass matrix $M_l^{(0)}$ and the neutrino mass matrix $M_\nu^{(0)}$. For simplicity we follow the same symmetry-breaking chain $M_l^{(0)} \rightarrow M_l^{(1)} \rightarrow M_l^{(2)}$ as discussed above (one of the six complex perturbations $\Delta M_l^{(2)}$ listed in Table 5.2 can be taken), but introduce the off-diagonal perturbations to $M_\nu^{(0)}$. To ensure the “maximal calculability” for the neutrino mass matrix, we require a special form of the overall perturbative matrix [84]: it consists solely of two unknown small parameters in addition to the scale parameter C_ν , and can be diagonalized by a *constant* orthogonal transformation independent of the neutrino masses. Then we are left with only three perturbative patterns satisfying these strong requirements:

$$M_\nu^{(2)} = C_\nu \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \varepsilon_\nu & 0 \\ \varepsilon_\nu & 0 & 0 \\ 0 & 0 & \delta_\nu \end{pmatrix} \right], \quad (5.16)$$

$$M_\nu^{(2)} = C_\nu \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \delta_\nu & 0 & 0 \\ 0 & 0 & \varepsilon_\nu \\ 0 & \varepsilon_\nu & 0 \end{pmatrix} \right], \quad (5.17)$$

$$M_\nu^{(2)} = C_\nu \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \varepsilon_\nu \\ 0 & \delta_\nu & 0 \\ \varepsilon_\nu & 0 & 0 \end{pmatrix} \right]. \quad (5.18)$$

Three forms of $M_\nu^{(2)}$ can be diagonalized by three Euler rotation matrices $R_{12}(\theta_{12})$, $R_{23}(\theta_{23})$ and $R_{31}(\theta_{31})$ (see also (3.25) for illustration), respectively, with a universal rotation angle $\theta_{ij} = 45^\circ$. In Ref. [85] the perturbative pattern (5.16) was discussed, but CP

violation was not taken into account. For each scenario of $M_\nu^{(2)}$, versus the given form of $M_l^{(2)}$, the resultant flavor mixing matrix reads as $V' = VR_{ij}$, where V has been obtained in (5.7). It is easy to check that only the pattern of $M_\nu^{(2)}$ in (5.16) can give rise to a sufficiently large mixing angle for the atmospheric neutrino oscillation. In this case the flavor mixing matrix $V' = VR_{12}$ with $\theta_{12} = 45^\circ$ takes the following form:

$$V' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} + i \xi_{V'} \sqrt{\frac{m_e}{m_\mu}} + \zeta_{V'} \frac{m_\mu}{m_\tau}, \quad (5.19)$$

where

$$\begin{aligned} \xi_{V'} &= \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \zeta_{V'} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}. \end{aligned} \quad (5.20)$$

Parametrizing V' in terms of three Euler angles and one CP -violating phase, as already done in (5.9), we arrive numerically at

$$\theta_l \approx 4^\circ, \quad \theta_\nu \approx 0^\circ, \quad \theta \approx 52^\circ, \quad \phi \approx 0^\circ. \quad (5.21)$$

Note that $\phi \approx 0^\circ$ is consistent with $\theta_\nu \approx 0$, i.e., the CP -violating phase ϕ can always be rotated away in the case that one flavor mixing angle vanishes. Therefore CP violation is absent, up to the given accuracy of V' . The smallness of $|V'_{e3}|$ ensures that the atmospheric and solar neutrino oscillations are approximately decoupled.

Let us calculate the mixing factors of solar and atmospheric neutrino oscillations with the help of (5.19). We obtain

$$\begin{aligned} \sin^2 2\theta_{\text{sun}} &= \frac{4}{3} \frac{m_e}{m_\mu}, \\ \sin^2 2\theta_{\text{atm}} &= \frac{8}{9} \left(1 + \frac{m_\mu}{m_\tau} \right) \end{aligned} \quad (5.22)$$

to a good degree of accuracy. Numerically $\sin^2 2\theta_{\text{sun}} \approx 0.0064$ and $\sin^2 2\theta_{\text{atm}} \approx 0.94$. Thus this flavor mixing scenario favors the small-angle MSW solution to the solar neutrino problem. Its consequence on the atmospheric neutrino oscillations is the same as the nearly bi-maximal mixing scenario discussed above.

If neutrinos are of the Majorana type, then the smallness of $|\theta_l|$ and $|\theta_\nu|$ in this small-versus-large mixing scenario implies that the effective mass factor of the neutrinoless double beta decay (i.e., $\langle m_\nu \rangle$) is dominated by m_1 . A strong constraint turns out to be $m_1 \leq 0.2$ eV, in view of current data on the $(\beta\beta)_{0\nu}$ decay. The sum of three neutrino masses has an upper bound of 0.6 eV, too small to account for the hot dark matter of the universe.

Bi-maximal mixing pattern Now let us take a look at the exactly “bi-maximal” mixing scenario of three neutrinos [78]. The relevant flavor mixing matrix, similar to the leading term of V in (5.7), reads as follows:

$$V'' = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (5.23)$$

This flavor mixing pattern is independent of charged lepton and neutrino masses, and it leads exactly to $\sin^2 2\theta_{\text{atm}} = \sin^2 2\theta_{\text{sun}} = 1$ for atmospheric and solar neutrino oscillations. Therefore it favors the “Just-So” solution (perhaps also the large-angle MSW solution) to the solar neutrino problem. We find that V'' can be derived from the following charged lepton and neutrino mass matrices [28]:

$$\begin{aligned} M_l &= \frac{C_l}{2} \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} \delta_l & 0 & 0 \\ 0 & 0 & \varepsilon_l \\ 0 & \varepsilon_l & 0 \end{pmatrix} \right], \\ M_\nu &= C_\nu \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \varepsilon_\nu & 0 \\ \varepsilon_\nu & 0 & 0 \\ 0 & 0 & \delta_\nu \end{pmatrix} \right], \end{aligned} \quad (5.24)$$

where $|\delta_{l,\nu}| \ll 1$ and $|\varepsilon_{l,\nu}| \ll 1$. In comparison with the democratic mass matrix $M_l^{(0)}$ given in (5.4), which is invariant under an $S(3)_L \times S(3)_R$ transformation, the matrix M_l in the limit $\delta_l = \varepsilon_l = 0$ only has an $S(2)_L \times S(2)_R$ symmetry. However M_ν in the limit $\delta_\nu = \varepsilon_\nu = 0$ takes the same form as $M_\nu^{(0)}$ in (5.4), which displays an $S(3)$ symmetry. The off-diagonal perturbation of M_l allows the masses of three charged leptons to be hierarchical:

$$\{m_e, m_\mu, m_\tau\} = \frac{C_l}{2} \{|\delta_l|, |\varepsilon_l|, 2 + \varepsilon_l\}. \quad (5.25)$$

One finds $C_l = m_\mu + m_\tau \approx 1.88 \text{ GeV}$, $|\varepsilon_l| = 2m_\mu/(m_\mu + m_\tau) \approx 0.11$ and $|\delta_l| = 2m_e/(m_\mu + m_\tau) \approx 5.4 \times 10^{-4}$. The off-diagonal perturbation of M_ν makes the three neutrino masses non-degenerate:

$$\{m_1, m_2, m_3\} = C_\nu \{1 + \varepsilon_\nu, 1 - \varepsilon_\nu, 1 + \delta_\nu\}. \quad (5.26)$$

For solar and atmospheric neutrino oscillations, one can take the mass-squared differences as in (2.21). We then arrive at $|\varepsilon_\nu|/|\delta_\nu| \approx \Delta m_{\text{sun}}^2/(2\Delta m_{\text{atm}}^2)$, of $\mathcal{O}(10^{-7})$ for the vacuum-oscillation solution and of $\mathcal{O}(10^{-3})$ to $\mathcal{O}(10^{-2})$ for the MSW solution to the solar neutrino problem. The diagonalization of M_l and M_ν leads to the flavor mixing matrix V'' . In Ref. [78] a different neutrino mass matrix has *reversely* been derived from the given V'' in the flavor basis where the charged lepton mass matrix is diagonal. The emergence of the bi-maximal mixing pattern from M_l and M_ν in (5.24) is obviously similar to that of the nearly bi-maximal mixing pattern obtained in (5.7), although the former is irrelevant to lepton masses and CP violation.

Note that three mixing angles of V'' are given as $\theta_l = 0$, $\theta_\nu = 45^\circ$ and $\theta = 45^\circ$. The vanishing θ_l leads to vanishing probabilities for $\nu_e \rightarrow \nu_\mu$ and $\nu_e \rightarrow \nu_\tau$ transitions in the long-baseline neutrino experiments at the atmospheric scale. This feature distinguishes the bi-maximal mixing ansatz from the nearly bi-maximal mixing ansatz.

If three neutrinos are of the Majorana type, then the near degeneracy of their masses implies that $\langle m_\nu \rangle \approx m_i$ for the neutrinoless double beta decay in the bi-maximal mixing scenario. Therefore $m_i < 0.2$ eV. The sum of three neutrino masses is insufficient to account for the hot dark matter of the universe.

Finally it is worth mentioning the so-called tri-maximal neutrino mixing scenario, in which the lepton flavor mixing matrix takes the form [86]

$$V''' = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad (5.27)$$

with ω a complex cube-root of unity. This mixing matrix could be derived from the permutation symmetry of the neutrino mass matrix in the flavor basis where the charged lepton mass matrix is diagonal. A special feature of V''' is that the rephasing-invariant measure of CP violation takes its maximal value, i.e., $\mathcal{J}_l = 1/(6\sqrt{3})$. In such a threefold maximal mixing scenario all survival and appearance probabilities of neutrino oscillations are universal, independent of the specific neutrino flavors. It is obvious that the plain tri-maximal neutrino mixing pattern is in conflict with current experimental data [18, 21]. If possible terrestrial matter effects are taken into account [87], there is a small chance that it might be reconciled with the experimental data.

In summary the idea of lepton flavor democracy and its explicit breaking can lead to phenomenologically favored flavor mixing patterns, yielding satisfactory interpretations of current experimental data on solar and atmospheric neutrino oscillations. Some theoretical attempts have recently been made towards deeper understanding of this interesting flavor symmetry (see, e.g., Refs. [80, 88]).

5.3 Seesaw-invariant texture of lepton mass matrices

If neutrinos are massive and lepton flavors are mixed, one may wonder whether there exist some similarities or relations between the mass and mixing textures of leptons and quarks. A basic question in understanding the fermion mass spectra is why the masses of three active neutrinos are so small compared with those of charged leptons or quarks. For the time being this question remains open, although a number of theoretical speculations towards a definite answer have been made. The smallness of chiral (left-handed) neutrino masses is perhaps attributed to the fact that they are electrically neutral fermions, or more exactly, to the Majorana feature of neutrino fields. In this picture the left-handed Majorana neutrinos can naturally acquire their masses through an effective seesaw mechanism [7]⁸:

$$M_\nu = (M_\nu^D)^T (M_R)^{-1} (M_\nu^D), \quad (5.28)$$

where M_ν^D and M_R denote Dirac and right-handed Majorana neutrino mass matrices, respectively. In most grand unified theories one takes $[M_\nu^D, M_u] = 0$, where M_u stands for the mass matrix of the up-type quark sector. In some left-right symmetric models one takes $[M_\nu^D, M_l] = 0$, where M_l is the charged lepton mass matrix. The mass matrix of

⁸For simplicity the notations of Majorana neutrino mass matrices used here are different from those used in (2.25). Obviously $M_\nu = M_\nu^M$ and $M_R = \tilde{M}_\nu^M$.

the heavy right-handed neutrinos (M_R) is practically unknown in almost all reasonable extensions of the standard model. For this reason a specific texture of M_R , in addition to that of M_ν^D , has to be assumed in order to calculate masses of the light left-handed Majorana neutrinos.

A phenomenologically favored texture of quark mass matrices has been presented in (4.23). The texture zeros in the (1,1), (1,3) and (3,1) positions of M_u and M_d could be the consequence of an underlying flavor symmetry. In the spirit of lepton-quark similarity we prescribe the same texture for the charged lepton and Dirac neutrino mass matrices:

$$\begin{aligned} M_l &= \begin{pmatrix} \mathbf{0} & D_l & \mathbf{0} \\ D_l^* & C_l & B_l \\ \mathbf{0} & B_l^* & A_l \end{pmatrix}, \\ M_\nu^D &= \begin{pmatrix} \mathbf{0} & D_\nu^D & \mathbf{0} \\ D_\nu^D & C_\nu^D & B_\nu^D \\ \mathbf{0} & B_\nu^D & A_\nu^D \end{pmatrix}, \end{aligned} \quad (5.29)$$

in which $|D_l| \ll |C_l| \sim |B_l| \ll |A_l|$ holds. Without loss of generality, we have taken M_ν^D to be real. The phase differences between M_l and M_ν^D are therefore denoted as $\varphi_1 \equiv \arg(D_l)$ and $\varphi_2 \equiv \arg(B_l)$. They are the source of leptonic CP violation in neutrino oscillations. Note that M_l can be decomposed into $M_l = P_l^\dagger \overline{M}_l P_l$, where \overline{M}_l is a real symmetric matrix with the same texture zeros as M_l , and $P_l = \text{Diag}\{1, e^{i\varphi_1}, e^{i(\varphi_1+\varphi_2)}\}$ is a Dirac-type phase matrix.

We conjecture that M_R could have the same texture as that of M_ν^D and M_l , i.e.,

$$M_R = \begin{pmatrix} \mathbf{0} & D_R & \mathbf{0} \\ D_R & C_R & B_R \\ \mathbf{0} & B_R & A_R \end{pmatrix}, \quad (5.30)$$

which might follow from a universal flavor symmetry hidden in the more fundamental theory of fermion mass generation. The matrix elements of M_R are in general complex. It is worth mentioning that the universal texture of M_R , M_ν^D (or M_u) and M_l can theoretically be obtained in the context of $SO(10)$ grand unified theories [73, 89]. Instead of exploring the theoretical details of these mass matrices, we proceed to calculate the left-handed neutrino mass matrix M_ν via the seesaw mechanism.

Given the textures of M_R and M_ν^D , it is straightforward to show that M_ν has the same texture:

$$M_\nu = \begin{pmatrix} \mathbf{0} & D_\nu & \mathbf{0} \\ D_\nu & C_\nu & B_\nu \\ \mathbf{0} & B_\nu & A_\nu \end{pmatrix}, \quad (5.31)$$

in which the four matrix elements are given by

$$\begin{aligned} A_\nu &= \frac{(A_\nu^D)^2}{A_R}, \\ B_\nu &= \frac{A_\nu^D B_\nu^D}{A_R} + \frac{B_\nu^D D_\nu^D}{D_R} - \frac{A_\nu^D D_\nu^D B_R}{A_R D_R}, \end{aligned}$$

$$\begin{aligned}
C_\nu &= \frac{(B_\nu^D)^2}{A_R} + \frac{2C_\nu^D D_\nu^D}{D_R} - \frac{(D_\nu^D)^2 C_R}{D_R^2} - \frac{2B_\nu^D D_\nu^D B_R}{A_R D_R} + \frac{(D_\nu^D)^2 B_R^2}{A_R D_R^2} , \\
D_\nu &= \frac{(D_\nu^D)^2}{D_R} .
\end{aligned} \tag{5.32}$$

Note that the texture zeros of M_ν^D and M_R manifest themselves again in M_ν , as a consequence of the seesaw mechanism. We refer this special texture of Dirac and Majorana neutrino mass matrices to the *seesaw-invariant* texture. As the matrix elements of M_R represent some kinds of superhigh energy scales [7] and those of M_ν^D amount to the mass scales of up-type quark masses ($[M_\nu^D, M_u] = 0$) or charged lepton masses ($[M_\nu^D, M_l] = 0$), a significant suppression of the matrix elements of M_ν is transparent in (5.32). This provides a natural interpretation of the smallness of the left-handed neutrino masses.

In practice, however, useful predictions for light neutrino masses and lepton flavor mixing angles are prevented due to the unknown parameters of M_R . A phenomenologically acceptable approach is to calculate the flavor mixing matrix starting directly from M_l and M_ν , regardless of the details of M_R . Subsequently we follow this strategy to examine how a neutrino mixing pattern with two large mixing angles can naturally emerge in the present model. For simplicity and instruction we assume that the symmetric neutrino mass matrix M_ν is real, therefore the only CP -violating phase existing in the lepton flavor mixing matrix is of the Dirac type and comes from the complex phases of M_l .

The real symmetric mass matrices \overline{M}_l and M_ν can be diagonalized by two orthogonal transformations:

$$\begin{aligned}
O_l^T \overline{M}_l O_l &= \text{Diag}\{-m_e, m_\mu, m_\tau\} , \\
O_\nu^T M_\nu O_\nu &= \text{Diag}\{-m_1, m_2, m_3\} .
\end{aligned} \tag{5.33}$$

In general the four parameters of \overline{M}_l (or M_ν) cannot be uniquely determined by inputting the measured mass eigenvalues. To obtain an analytically simple solution for O_l or O_ν , we typically specify $C_l = m_\mu$ and $C_\nu = m_2$ (the similar choices $C_u = m_c$ and $C_d = m_s$ are favored for the quark mass matrices M_u and M_d in (4.23) to reproduce the experimental values of quark flavor mixing angles [8, 51]). Then the other parameters of \overline{M}_l and M_μ can be determined in terms of the charged lepton and neutrino masses, respectively:

$$\begin{aligned}
A_l &= m_\tau - m_e , \\
|B_l| &= \left[\frac{m_e m_\tau (m_\tau - m_e - m_\mu)}{m_\tau - m_e} \right]^{1/2} , \\
|D_l| &= \left(\frac{m_e m_\mu m_\tau}{m_\tau - m_e} \right)^{1/2} ;
\end{aligned} \tag{5.34}$$

and

$$\begin{aligned}
A_\nu &= m_3 - m_1 , \\
B_\nu &= \left[\frac{m_1 m_3 (m_3 - m_1 - m_2)}{m_3 - m_1} \right]^{1/2} , \\
D_\nu &= \left(\frac{m_1 m_2 m_3}{m_3 - m_1} \right)^{1/2} .
\end{aligned} \tag{5.35}$$

In this special but interesting case, we obtain the matrix elements of O_l and O_ν in terms of the mass ratios $x_l \equiv m_e/m_\mu$, $y_l \equiv m_\mu/m_\tau$ and $x_\nu \equiv m_1/m_2$ and $y_\nu \equiv m_2/m_3$ as follows (the subscripts “ l ” and “ ν ” are neglected for simplicity):

$$\begin{aligned}
O_{11} &= + \left[\frac{1}{(1+x)(1-x^2y^2)} \right]^{1/2}, \\
O_{12} &= + \left[\frac{x(1-y-xy)}{(1+x)(1-y)(1-xy)} \right]^{1/2}, \\
O_{13} &= + \left[\frac{x^2y^3}{(1-y)(1-x^2y^2)} \right]^{1/2}, \\
O_{21} &= - \left[\frac{x}{(1+x)(1+xy)} \right]^{1/2}, \\
O_{22} &= + \left[\frac{1-y-xy}{(1+x)(1-y)} \right]^{1/2}, \\
O_{23} &= + \left[\frac{xy}{(1-y)(1+xy)} \right]^{1/2}, \\
O_{31} &= + \left[\frac{x^2y(1-y-xy)}{(1+x)(1-x^2y^2)} \right]^{1/2}, \\
O_{32} &= - \left[\frac{xy}{(1+x)(1-y)(1-xy)} \right]^{1/2}, \\
O_{33} &= + \left[\frac{1-y-xy}{(1-y)(1-x^2y^2)} \right]^{1/2}.
\end{aligned} \tag{5.36}$$

It is easy to check that O_l is very close to the unity matrix, because of the smallness of x_l ($= 0.0048$) and y_l ($= 0.0594$) [1]. Instead O_ν can significantly deviate from the unity matrix, if three neutrino masses do not have a strong hierarchy.

The leptonic flavor mixing matrix $V \equiv O_l^T P_l O_\nu$ links the neutrino mass eigenstates (ν_1, ν_2, ν_3) to the neutrino flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$. Assuming the phase differences between M_l and M_ν^D to have the special values $\varphi_1 = \pi/2$ and $\varphi_2 = \pi$ (a case favored for the quark mass matrices in (4.23) to reproduce the measured flavor mixing angles and CP violation [8, 47]), we obtain the magnitudes of $|V_{i\alpha}|^2$ as follows:

$$|V_{\alpha i}|^2 = \left(O_{1\alpha}^l O_{1i}^\nu \right)^2 + \left(O_{2\alpha}^l O_{2i}^\nu - O_{3\alpha}^l O_{3i}^\nu \right)^2, \tag{5.37}$$

where α runs over (e, μ, τ) and i over $(1, 2, 3)$. The rephasing-invariant measure of CP violation, \mathcal{J}_l , can then be calculated by use of the formula given in (3.12). As one can see from (5.36), $|V_{i\alpha}|$ and \mathcal{J}_l are functions of the neutrino mass ratios x_ν and y_ν .

Without loss of much generality we take $x_\nu \leq 1$ and $y_\nu \leq 1$. Taking (2.21) into account, we obtain the ratio of Δm_{sun}^2 to Δm_{atm}^2 in terms of x_ν and y_ν :

$$R \equiv \frac{\Delta m_{\text{sun}}^2}{\Delta m_{\text{atm}}^2} = y_\nu^2 \frac{1-x_\nu^2}{1-y_\nu^2} \ll 1. \tag{5.38}$$

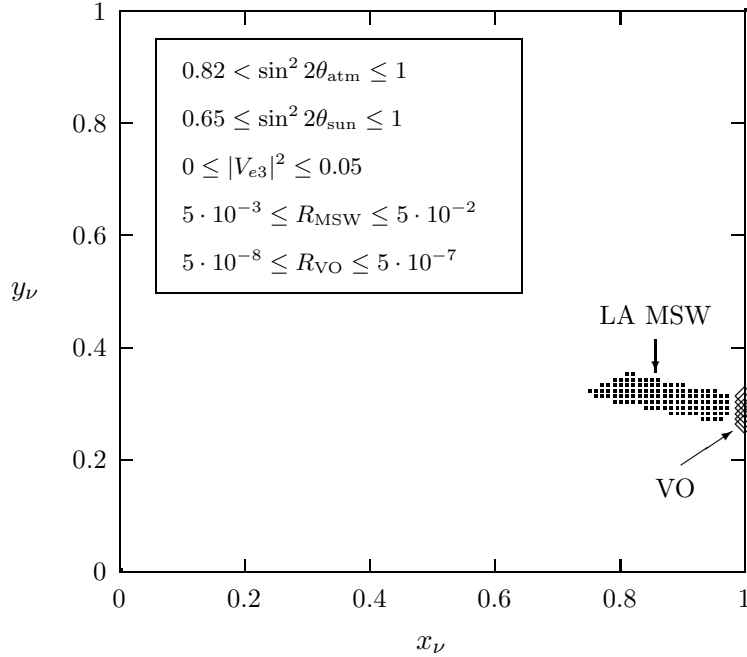


Figure 5.1: Allowed ranges of the neutrino mass ratios $x_\nu \equiv m_1/m_2$ and $y_\nu \equiv m_2/m_3$ constrained by current data, where $R \equiv \Delta m_{\text{sun}}^2/\Delta m_{\text{atm}}^2$ taking different values for the large-angle MSW and vacuum oscillation solutions to the solar neutrino problem.

Note that $R \ll 1$ is imposed by current atmospheric and solar neutrino oscillation data. This condition is satisfied if both $x_\nu \ll 1$ and $y_\nu \ll 1$ hold, or if $x_\nu \approx 1$ and $y_\nu < x_\nu$ hold for proper values of y_ν . For vacuum oscillation and large-angle MSW solutions to the solar neutrino problem, one obtains $R_{\text{VO}} \sim 10^{-7}$ and $R_{\text{MSW}} \sim 10^{-2}$ respectively⁹. The mixing factors of solar ($\nu_e \rightarrow \nu_e$ disappearance) and atmospheric ($\nu_\mu \rightarrow \nu_\mu$ disappearance) neutrino oscillations can be given, under the approximate decoupling condition $|V_{e3}|^2 \ll 1$, as follows:

$$\begin{aligned} \sin^2 2\theta_{\text{sun}} &= 4 |V_{e1}|^2 |V_{e2}|^2, \\ \sin^2 2\theta_{\text{atm}} &= 4 |V_{\mu 3}|^2 (1 - |V_{\mu 3}|^2). \end{aligned} \quad (5.39)$$

Clearly R , $|V_{e3}|^2$, $\sin^2 2\theta_{\text{sun}}$ and $\sin^2 2\theta_{\text{atm}}$ are all functions of two independent neutrino mass ratios. We are therefore able to find out the allowed parameter space for (x_ν, y_ν) by fitting current experimental data on R , $|V_{e3}|^2$, and so on. Explicitly we take $0.82 < \sin^2 2\theta_{\text{atm}} \leq 1$, $0.65 \leq \sin^2 2\theta_{\text{sun}} \leq 1$, and $|V_{e3}|^2 \leq 0.05$ [20]. In addition, $5 \times 10^{-3} \leq R_{\text{MSW}} \leq 5 \times 10^{-2}$ and $5 \times 10^{-8} \leq R_{\text{VO}} \leq 5 \times 10^{-7}$ are adopted. With these inputs (and with $x_l = 0.00484$ and $y_l = 0.0594$ [1]), we carry out a numerical calculation based on the formulae given in (5.36) – (5.39). The allowed ranges of x_ν and y_ν are then illustrated in Fig. 5.1, without any fine tuning. We observe that $y_\nu \sim 0.3$ for either the large-angle

⁹Here we do not take the small-angle MSW solution into account. Indeed a numerical study shows that this solution is not compatible with the model under discussion.

MSW solution or the vacuum oscillation solution. The former requires $x_\nu \geq 0.8$, while the latter requires $x_\nu \approx 1$. The magnitude of \mathcal{J}_l is in the range $(7-8) \times 10^{-3}$, implying that a moderate strength of CP violation is allowed by the model.

We see that this lepton mixing scenario can well accommodate the large-angle MSW oscillations of solar neutrinos. It is also compatible with the vacuum oscillation solution to the solar neutrino problem, although the allowed parameter space is much smaller. In both cases the atmospheric neutrino oscillation with a large mixing factor can be incorporated. Note that the results $0.8 \leq x_\nu < 1$ and $y_\nu \sim 0.3$ indicate a nearly degenerate neutrino mass spectrum, in contrast with the strong mass hierarchy of charged leptons. This is not a surprise, because it is technically difficult to obtain a lepton mixing pattern with two large mixing angles, when three neutrino masses are hierarchical. The near degeneracy of three neutrino masses in our model implies that they may be the proper candidate for the hot dark matter of the universe (in this case $m_1 + m_2 + m_3 \sim 5$ to 10 eV is required). Of course such a neutrino mass spectrum may have no conflict with the stringent upper limit on the effective mass factor $\langle m_\nu \rangle$ of the neutrinoless double beta decay [17]. The reason is simply that cancellations can take place in the expression of $\langle m_\nu \rangle$, in particular when CP -violating phases of the Majorana type are introduced.

Once the neutrino masses are determined in the future neutrino experiments, one may specify the form of M_ν in a reverse way. The right-handed neutrino mass matrix M_R can then be obtained via the seesaw mechanism, if $[M_\nu^D, M_u] = 0$ or $[M_\nu^D, M_l] = 0$ is taken. In an explicit theoretical framework (e.g., the $SO(10)$ grand unified model) the energy scale, at which the universal texture of M_u , M_l , M_R and M_ν holds, should be specified; and the running effects of these mass matrices between the superhigh and low energy scales should be taken into account. To make an order-of-magnitude estimate of M_R , we take $M_\nu^D = M_u$ and $m_1 \approx m_2 \approx m_3/3 \approx 1.5$ eV, and neglect all possible scale-dependent running effects (see section 5.5 for a detailed discussion). With the help of (5.32), (5.34) and (5.35), we obtain

$$\begin{aligned} M_u &\sim \begin{pmatrix} \mathbf{0} & \mathcal{O}(10^{-2}) & \mathbf{0} \\ \mathcal{O}(10^{-2}) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathbf{0} & \mathcal{O}(1) & \mathcal{O}(10^2) \end{pmatrix}, \\ M_R &\sim \begin{pmatrix} \mathbf{0} & \mathcal{O}(10^6) & \mathbf{0} \\ \mathcal{O}(10^6) & \mathcal{O}(10^8) & \mathcal{O}(10^{10}) \\ \mathbf{0} & \mathcal{O}(10^{10}) & \mathcal{O}(10^{13}) \end{pmatrix} \end{aligned} \quad (5.40)$$

in units of GeV; and

$$M_\nu \sim \begin{pmatrix} 0 & \mathcal{O}(1) & 0 \\ \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) \\ 0 & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} \quad (5.41)$$

in units of eV. This simple example illustrates that the tiny and nearly degenerate masses of left-handed neutrinos can naturally come forth from the hierarchical textures of M_ν^D and M_R through the seesaw mechanism.

Without the assumptions $C_l = m_\mu$ and $C_\nu = m_2$ as well as $\varphi_1 = \pi/2$ and $\varphi_2 = \pi$, one can find out other possible parameter spaces of M_l and M_ν to accommodate the atmospheric neutrino oscillation data and allow the small-angle MSW solution, the large-angle

MSW solution or the vacuum oscillation solution to the solar neutrino problem. A strong correlation between the mixing factors of solar and atmospheric neutrino oscillations in the model is generally expected, unlike those (nearly) bi-maximal or tri-maximal neutrino mixing scenarios in which the lepton mixing angles are essentially algebraic numbers independent of the neutrino mass ratios.

5.4 An illustrative scheme of four-neutrino mixing

In the context of three neutrino species, we have discussed several instructive schemes of lepton mass matrices which can interpret current experimental data on atmospheric and solar neutrino oscillations. The LSND evidence for $\nu_\mu \rightarrow \nu_e$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ oscillations have so far been put aside. To make a simultaneous interpretation of solar, atmospheric and LSND neutrino oscillation data, which have three different levels of neutrino mass-squared differences (i.e., $\Delta m_{\text{sun}}^2 \sim 10^{-10} - 10^{-5} \text{ eV}^2$, $\Delta m_{\text{atm}}^2 \sim 10^{-3} \text{ eV}^2$ and $\Delta m_{\text{LSND}}^2 \sim 1 \text{ eV}^2$), one has to go beyond the conventional three-neutrino framework by incorporating one sterile light neutrino (ν_s with mass m_0). As already pointed out in section 2.2, a particularly favored four-neutrino mixing scenario is that the $\nu_e \rightarrow \nu_s$, $\nu_\mu \rightarrow \nu_\tau$ and $\nu_\mu \rightarrow \nu_e$ transitions are respectively responsible for the solar, atmospheric and LSND neutrino oscillations. Let us denote the mass eigenstates of ν_s , ν_e , ν_μ and ν_τ as ν_0 , ν_1 , ν_2 and ν_3 , respectively. Then the 4×4 flavor mixing matrix V is explicitly given as

$$\begin{pmatrix} \nu_s \\ \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} V_{s0} & V_{s1} & V_{s2} & V_{s3} \\ V_{e0} & V_{e1} & V_{e2} & V_{e3} \\ V_{\mu 0} & V_{\mu 1} & V_{\mu 2} & V_{\mu 3} \\ V_{\tau 0} & V_{\tau 1} & V_{\tau 2} & V_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_0 \\ \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \quad (5.42)$$

Note that the four-neutrino mass spectrum is characterized by

$$\begin{aligned} \Delta m_{\text{sun}}^2 &= \Delta m_{10}^2 \equiv |m_1^2 - m_0^2|, \\ \Delta m_{\text{atm}}^2 &= \Delta m_{32}^2 \equiv |m_3^2 - m_2^2|, \\ \Delta m_{\text{LSND}}^2 &= \Delta m_{21}^2 \equiv |m_2^2 - m_1^2|. \end{aligned} \quad (5.43)$$

In contrast, the mass spectrum of charged leptons remains unchanged. It should be noted that the four-neutrino mixing phenomenon is not solely the property of a 4×4 neutrino mass matrix. Indeed the realistic lepton flavor mixing arises from the mismatch between diagonalizing the charged lepton and neutrino mass matrices in a given flavor basis. Enlarging the number of charged lepton families to four might pose a problem, because the fourth charged lepton must be sufficiently heavy (with mass $> 80 \text{ GeV}$ [1]) and the existence of such a heavy fermion requires credible theoretical motivation and experimental evidence. To avoid this “fourth charged lepton” problem and the related complexity, most authors have chosen to construct the four-neutrino mixing scenarios in the flavor basis in which the charged lepton mass matrix is diagonal. Such a treatment is of course not perfect, but it may serve as an acceptable approach to illustrate some features of the four-neutrino mixing phenomenon. We therefore follow the same strategy in the following.

For simplicity we assume CP invariance. In the absence of CP violation a generic 4×4 (Dirac or Majorana) neutrino mass matrix M_ν consists of 10 arbitrary parameters. Some of these entries receive strong restrictions from current experimental data and may be negligibly small in magnitude. In the flavor basis $(\nu_s, \nu_e, \nu_\mu, \nu_\tau)$ one can diagonalize M_ν through an orthogonal transformation:

$$V^T M_\nu V = \text{Diag}\{m_0, m_1, m_2, m_3\}, \quad (5.44)$$

where the real lepton flavor mixing matrix V links the neutrino mass eigenstates $(\nu_0, \nu_1, \nu_2 \text{ and } \nu_3)$ to the neutrino flavor eigenstates $(\nu_s, \nu_e, \nu_\mu \text{ and } \nu_\tau)$. Note that the strength of active-sterile neutrino mixing is strictly constrained by astrophysics and cosmology. A careful analysis of recent astrophysical data yields an upper limit $N_\nu^{\text{BBN}} \leq 3.2$ (at the 95% confidence level [90]) for the effective number of neutrinos in Big-Bang nucleosynthesis, although the matter remains quite controversial [91]. This upper bound implies that atmospheric $(\nu_\mu \rightarrow \nu_\tau)$ and solar $(\nu_e \rightarrow \nu_s)$ neutrino oscillations are essentially decoupled in the framework of four-neutrino mixing. As a consequence, we arrive at the following leading order result [92]

$$V \approx \begin{pmatrix} c_\odot & -s_\odot & 0 & 0 \\ s_\odot & c_\odot & 0 & 0 \\ 0 & 0 & c_\bullet & s_\bullet \\ 0 & 0 & -s_\bullet & c_\bullet \end{pmatrix}, \quad (5.45)$$

where $s_\odot \equiv \sin \theta_{\text{sun}}$, $c_\bullet \equiv \cos \theta_{\text{atm}}$, and so on. To accommodate the LSND neutrino oscillation data (i.e., $\nu_\mu \rightarrow \nu_e$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ transitions with a small mixing factor $\sin^2 2\theta_{\text{LSND}} \sim 10^{-3} - 10^{-2}$), a small admixture between (ν_e, ν_μ) and (ν_1, ν_2) states needs to be introduced. We therefore rotate the ν_1 - ν_2 sector of V by a very small angle $\epsilon \sim \theta_{\text{LSND}}$. Such a rotation can be done from either the left-handed side of V or the right-handed side of V . The resultant flavor mixing matrix V' deviates only slightly from V :

$$V' \approx \begin{pmatrix} c_\odot & -s_\odot & -\epsilon s_\odot & 0 \\ s_\odot & c_\odot & \epsilon c_\odot & 0 \\ 0 & -\epsilon c_\bullet & c_\bullet & s_\bullet \\ 0 & \epsilon s_\bullet & -s_\bullet & c_\bullet \end{pmatrix}, \quad (5.46)$$

or

$$V' \approx \begin{pmatrix} c_\odot & -s_\odot & 0 & 0 \\ s_\odot & c_\odot & \epsilon c_\bullet & \epsilon s_\bullet \\ -\epsilon s_\odot & -\epsilon c_\odot & c_\bullet & s_\bullet \\ 0 & 0 & -s_\bullet & c_\bullet \end{pmatrix}. \quad (5.47)$$

At this stage the texture of the neutrino mass matrix M_ν can straightforwardly be recast from its mass eigenvalues and V' , as shown in (5.3).

The four-neutrino mixing pattern V' with $\theta_{\text{atm}} \approx 45^\circ$ may generally arise from the following neutrino mass matrix [25, 93]:

$$M_\nu = m \begin{pmatrix} \delta_1 & \delta_2 & 0 & 0 \\ \delta_2 & 0 & 0 & \delta_3 \\ 0 & 0 & \delta_4 & 1 \\ 0 & \delta_3 & 1 & \pm \delta_4 \end{pmatrix}, \quad (5.48)$$

where m measures the mass scale of the LSND neutrino oscillations (i.e., $\Delta m_{\text{LSND}}^2 \approx m^2$), and the other dimensionless parameters satisfy $|\delta_i| \ll 1$ (for $i = 1, \dots, 4$). We obtain $\tan 2\theta_{\text{sun}} \approx 2|\delta_2/\delta_1|$ and $\epsilon \approx |\delta_3|$, in addition to $\theta_{\text{atm}} \approx 45^\circ$. The relative magnitudes of δ_i can properly be chosen in order to describe the small-angle MSW solution ($|\delta_2| \ll |\delta_1|$; $|\delta_4| \ll |\delta_3| \ll 1$), the large-angle MSW solution ($|\delta_1|, |\delta_2|, |\delta_4| \ll |\delta_3| \ll 1$), or the vacuum oscillation solution ($|\delta_1| \ll |\delta_2| \ll |\delta_4| \ll |\delta_3| \ll 1$) to the solar neutrino problem [25]. In any case the desired mass hierarchy $m_0, m_1 \ll m_2, m_3$ as illustrated in Fig. 2.1(a) appears; i.e., the neutrino mass spectrum is characterized by two nearly degenerate mass pairs separated by a mass gap of $m \sim 1$ eV.

The consequences of this simple four-neutrino mixing scenario on the long-baseline neutrino experiments have been explored in Ref. [25] in some detail. Also a theoretical attempt towards understanding such a specific texture of M_ν has been made in Ref. [93]. One can certainly find out some other interesting textures of the 4×4 neutrino mass matrix to accommodate the present solar, atmospheric and LSND neutrino oscillation data [94]. Whether there exists such a light sterile neutrino in nature remains an open question and could be resolved by the future neutrino experiments (see, e.g., Ref. [95] for a comprehensive discussion). On the theoretical side, the sterile neutrino can be introduced as an extra fermion to the standard model and its mass can be suppressed by use of the radiative mechanism. It is also possible to double the particle spectrum and the gauge forces of the standard model by including a mirror sector to it and inducing the Majorana neutrino masses via non-renormalizable operators. In this framework the lightest neutrino of the mirror sector will be identified as the sterile neutrino. For a review of the relevant theoretical models of sterile neutrinos, we refer the reader to Ref. [96].

5.5 Scale dependence of the neutrino mass matrix

So far we have discussed some instructive scenarios of lepton mass matrices at low energy scales, at which their consequences on neutrino oscillations can directly be confronted with the experimental data. From the theoretical point of view, however, a phenomenologically favored texture of lepton mass matrices might only serve as the low-scale approximation of a more fundamental model responsible for the fermion mass generation at superhigh energy scales. It is therefore desirable to investigate the scale dependence of lepton mass matrices, as that of quark mass matrices, by use of the renormalization-group equations. The running behaviors of the neutrino mass matrix is, however, strongly model-dependent. To be specific we only consider the possibility that the light Majorana neutrino masses are generated by the conventional seesaw mechanism with a singlet-neutrino mass scale $M_0 \sim 10^{13}$ GeV [97]¹⁰. Below this mass scale the Yukawa couplings of Dirac neutrinos become decoupled. Thus the running of the left-handed neutrino mass matrix M_ν with the energy scale μ can be described, in the framework of the minimal supersymmetric standard model, by [98]

$$16\pi^2 \frac{dM_\nu}{d\chi} = \left[-\left(\frac{6}{5}g_1^2 + 6g_2^2\right) + 6\text{Tr}\left(Y_u Y_u^\dagger\right) \right] M_\nu + \left(Y_l Y_l^\dagger\right) M_\nu + M_\nu \left(Y_l Y_l^\dagger\right)^T, \quad (5.49)$$

¹⁰Certainly the heavy Majorana neutrino masses are unnecessary to be degenerate, and the structure of M_R may have non-negligible effects on the renormalization-group equations of M_ν . For the purpose of illustration we simply assume that M_R is characterized by a single mass scale M_0 .

where $\chi = \ln(\mu/M_0)$; g_i (for $i = 1$ and 2) denote the gauge couplings; Y_u and Y_l are the Yukawa coupling matrices of the up-type quarks and the charged leptons, respectively.

In the flavor basis where the charged lepton mass matrix is diagonal, one can simplify the renormalization-group equation (5.49) and obtain the running behaviors of individual elements of M_ν at the weak scale $\mu = M_Z$. Neglecting the tiny contributions from up and charm quarks and defining the evolution functions

$$\begin{aligned} I_g &= \exp \left[+\frac{1}{16\pi^2} \int_0^{\ln(M_0/M_Z)} \left(\frac{6}{5}g_1^2(\chi) + 6g_2^2(\chi) \right) d\chi \right] , \\ I_t &= \exp \left[-\frac{1}{16\pi^2} \int_0^{\ln(M_0/M_Z)} f_t^2(\chi) d\chi \right] , \\ I_\alpha &= \exp \left[-\frac{1}{16\pi^2} \int_0^{\ln(M_0/M_Z)} f_\alpha^2(\chi) d\chi \right] , \end{aligned} \quad (5.50)$$

where f_t and f_α (for $\alpha = e, \mu, \tau$) are the Yukawa coupling eigenvalues of the top quark and charged leptons, one can solve (5.49) and arrive at

$$M_\nu(M_Z) = \left(I_g I_t^6 \right) T_l \cdot M_\nu(M_0) \cdot T_l \quad (5.51)$$

with $T_l = \text{Diag}\{I_e, I_\mu, I_\tau\}$. The overall factor $(I_g I_t^6)$ in (5.51) does not affect the relative magnitudes of the matrix elements of M_ν . Only the matrix T_l , which amounts to the unity matrix at the mass scale M_0 , can modify the texture of the light neutrino mass matrix from M_0 to M_Z . The mass hierarchy of three charged leptons (i.e., $f_e < f_\mu < f_\tau$) implies $I_e > I_\mu > I_\tau$ at any energy scale below M_0 . If three neutrinos were exactly degenerate (i.e., $m_1 = m_2 = m_3$) at the scale M_0 , they would become non-degenerate and have the spectrum $m_1 > m_2 > m_3$ at a lower scale like M_Z . Indeed the magnitude of I_τ may substantially deviate from unity, if $\tan \beta_{\text{susy}}$ (the ratio of Higgs vacuum expectation values) takes large values. In contrast, $I_e \approx I_\mu \approx 1$ is expected to be a good approximation.

To be more explicit, we take $m_t(M_Z) = 175$ GeV, $m_b(M_Z) = 2.9$ GeV, $m_\tau(M_Z) = 1.777$ GeV, $g_1^2(M_Z) = 0.127$ and $g_2^2(M_Z) = 0.42$ to calculate the evolution factors I_g , I_t and I_τ in the framework of the minimal supersymmetric standard model. We obtain $I_g \approx 1.6$. The magnitudes of I_t and I_τ depend on the input value of $\tan \beta_{\text{susy}}$. For example, $I_t \approx 0.89$ and $I_\tau \approx 1.0$ for $\tan \beta_{\text{susy}} = 10$; while $I_t \approx 0.86$ and $I_\tau \approx 0.9$ for $\tan \beta_{\text{susy}} = 60$. We see that the overall factor of $M_\nu(M_Z)$ takes the value $(I_g I_t^6) \approx 0.80$ and 0.65 , respectively, for $\tan \beta_{\text{susy}} = 10$ and 60 . The specific texture of $M_\nu(M_Z)$ is different from that of $M_\nu(M_0)$, only because of the deviation of I_τ from unity.

For illustration we take the general symmetric texture of $M_\nu(M_0)$ in the basis where $M_l(M_0)$ is diagonal:

$$M_\nu(M_0) = \begin{pmatrix} E_\nu(M_0) & D_\nu(M_0) & F_\nu(M_0) \\ D_\nu(M_0) & C_\nu(M_0) & B_\nu(M_0) \\ F_\nu(M_0) & B_\nu(M_0) & A_\nu(M_0) \end{pmatrix} . \quad (5.52)$$

With the help of (5.51), we obtain the renormalized neutrino mass matrix at the scale M_Z

as follows:

$$M_\nu(M_Z) = \begin{pmatrix} E_\nu(M_Z) & D_\nu(M_Z) & F_\nu(M_Z) \\ D_\nu(M_Z) & C_\nu(M_Z) & B_\nu(M_Z) \\ F_\nu(M_Z) & B_\nu(M_Z) & A_\nu(M_Z) \end{pmatrix}, \quad (5.53)$$

where

$$\begin{aligned} A_\nu(M_Z) &= (I_g I_t^6) I_\tau^2 A_\nu(M_0), \\ B_\nu(M_Z) &= (I_g I_t^6) I_\mu I_\tau B_\nu(M_0), \\ C_\nu(M_Z) &= (I_g I_t^6) I_\mu^2 C_\nu(M_0), \\ D_\nu(M_Z) &= (I_g I_t^6) I_e I_\mu D_\nu(M_0), \\ E_\nu(M_Z) &= (I_g I_t^6) I_e^2 E_\nu(M_0), \\ F_\nu(M_Z) &= (I_g I_t^6) I_e I_\tau F_\nu(M_0). \end{aligned} \quad (5.54)$$

An interesting feature of $M_\nu(M_Z)$ would be that its possible texture zeros remain the same as those of $M_\nu(M_0)$, up to the degree of accuracy taken in deriving (5.51). This result could in some sense support our conjecture made in section 5.3; i.e., the seesaw-invariant texture of lepton mass matrices, which might arise from a dynamical mechanism of fermion mass generation, holds essentially at both superhigh and low energy scales.

Of course the evolution of M_ν from M_0 to M_Z will in general affect the form of the lepton flavor mixing matrix $V(M_0)$. This is naturally expected, as the flavor mixing angles are sensitive to the off-diagonal elements of the neutrino mass matrix. Therefore the mixing pattern $V(M_Z)$, in particular its (ν_μ, ν_τ) and (ν_e, ν_τ) sectors, might depart significantly from $V(M_0)$ if I_τ deviates substantially from unity. The explicit running behaviors of the flavor mixing matrix elements are, however, correlated in a complicated way with those of the neutrino mass eigenvalues (see, e.g., Ref. [99] for detailed calculations of $V(M_Z)$ from $V(M_0)$ with different spectra of neutrino masses).

It is possible to invoke another model in which the light Majorana neutrino masses are simply generated by a new non-renormalizable interaction with dimension-5 mass operators at the scale $\Lambda \sim 10^5$ GeV. In this case there is no Dirac neutrino coupling to be renormalized [97], thus the running effect of M_ν below the scale Λ can be described by the same formulae given above. As $\ln(\Lambda/M_Z) \sim 7$, the deviation of I_τ (as well as I_t) from unity is insignificant and even negligible. Then we are left with $M_\nu(M_Z) \approx M_\nu(\Lambda)$ and $V(M_Z) \approx V(\Lambda)$.

5.6 CP violation in long-baseline neutrino experiments

It is known that within the framework of three lepton families the strength of CP violation is universal in $\nu_e \rightarrow \nu_\mu$, $\nu_\tau \rightarrow \nu_e$ and $\nu_\tau \rightarrow \nu_\mu$ transitions (see Ref. [100] as well as (3.22) and (3.23) for details). The asymmetry between the probabilities of two CP -conjugate appearance processes is uniquely given as

$$\begin{aligned} \Delta_{CP} &= P(\nu_\alpha \rightarrow \nu_\beta) - P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta) \\ &= -16 \mathcal{J}_l \sin F_{\text{sun}} \sin^2 F_{\text{atm}}, \end{aligned} \quad (5.55)$$

where $(\alpha, \beta) = (e, \mu)$, (μ, τ) or (τ, e) ; \mathcal{J}_l is the universal CP -violating parameter; F_{sun} and F_{atm} measure the frequencies of solar and atmospheric neutrino oscillations, respectively. The T -violating asymmetry can be obtained in a similar way:

$$\begin{aligned}\Delta_T &= P(\nu_\alpha \rightarrow \nu_\beta) - P(\nu_\beta \rightarrow \nu_\alpha) \\ &= -16\mathcal{J}_l \sin F_{\text{sun}} \sin^2 F_{\text{atm}} , \\ \bar{\Delta}_T &= P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta) - P(\bar{\nu}_\beta \rightarrow \bar{\nu}_\alpha) \\ &= +16\mathcal{J}_l \sin F_{\text{sun}} \sin^2 F_{\text{atm}} .\end{aligned}\tag{5.56}$$

These formulas show clearly that CP or T violation is a property of all three lepton families. The relationship $\Delta_T = \Delta_{CP}$ is a straightforward consequence of CPT invariance, and $\bar{\Delta}_T = -\Delta_T$ indicates that the T -violating measurable is an odd function of time.

The CP - and T -violating signals can in principle be measured in the long-baseline neutrino experiments [101]. As Δ_{CP} and Δ_T depend linearly on the oscillation term $\sin F_{\text{sun}}$, the length of the baseline suitable for detecting CP and T asymmetries should satisfy the condition $L \sim E/\Delta m_{\text{sun}}^2$. This requirement singles out the large-angle MSW solution, which has $\Delta m_{\text{sun}}^2 \sim 10^{-5}$ to 10^{-4} eV² and $\sin^2 2\theta_{\text{sun}} \sim 0.65$ to 1 [20], among three possible solutions to the solar neutrino problem. The small-angle MSW solution poses a problem, because it does not give rise to a relatively large magnitude of \mathcal{J}_l , which determines the significance of practical CP - or T -violating signals. The long wave-length vacuum oscillation solution requires $\Delta m_{\text{sun}}^2 \sim 10^{-10}$ eV², too small to meet the realistic long-baseline prerequisite.

The observation of Δ_T might be free from the matter effects of the earth, which is possible to fake the genuine CP asymmetry Δ_{CP} in any long-baseline neutrino experiment. The joint measurement of $\nu_\alpha \rightarrow \nu_\beta$ and $\nu_\beta \rightarrow \nu_\alpha$ transitions to determine Δ_T is, however, a challenging task in practice. Probably it could only be realized in a neutrino factory, where high-quality neutrino beams can be produced with high-intensity muon storage rings [102].

To illustrate the earth-induced matter effects, we write out the effective Hamiltonians for neutrinos and antineutrinos [40]:

$$\begin{aligned}\mathcal{H}_\nu &= \frac{1}{2E} \left[V \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} V^\dagger + \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] , \\ \mathcal{H}_{\bar{\nu}} &= \frac{1}{2E} \left[V^* \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} V^T - \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] ,\end{aligned}\tag{5.57}$$

where V and m_i (for $i = 1, 2, 3$) denote the flavor mixing matrix and neutrino mass eigenvalues, respectively, in vacuum; and $a = 2\sqrt{2}G_F N_e E \sim \mathcal{O}(10^{-4})$ eV² $\cdot E/[\text{GeV}]$ describes the charged current interaction with electrons (N_e and E stand for the background density of electrons and the neutrino energy, respectively). Once $a \sim \Delta m_{21}^2$ or Δm_{32}^2 for given values of E , significant matter effects will enter neutrino oscillations. In this case the diagonalization of \mathcal{H}_ν and $\mathcal{H}_{\bar{\nu}}$ by two different unitary matrices leads to the *effective*

neutrino mass eigenvalues which deviate somehow from the genuine ones. Moreover the opposite signs of a in \mathcal{H}_ν and $\mathcal{H}_{\bar{\nu}}$ signify that the background matter is not CP invariant. This provokes a fake CP asymmetry between neutrino and antineutrino transitions, whose magnitude could in some cases be comparable with or dominant over the genuine CP asymmetry measured by \mathcal{J}_l . Roughly speaking, the longer the baseline, which in turn requires the higher neutrino beam energy, the larger the matter effect on leptonic CP violation [101].

In practical experiments one might prefer to measure the CP asymmetry Δ_{CP} normalized by the sum of two CP -conjugate transition probabilities. Such a signal is particularly significant for $\nu_\mu \rightarrow \nu_e$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ oscillations, because their CP -conserving and CP -violating parts are comparable in magnitude (see (3.23) for illustration). To be specific we adopt the nearly bi-maximal neutrino mixing scenario introduced in section 5.2 and calculate the CP -violating observable in vacuum [39]:

$$\begin{aligned}
\mathcal{A} &= \frac{P(\nu_\mu \rightarrow \nu_e) - P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e)}{P(\nu_\mu \rightarrow \nu_e) + P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e)} \\
&= \frac{P(\nu_\mu \rightarrow \nu_e) - P(\nu_e \rightarrow \nu_\mu)}{P(\nu_\mu \rightarrow \nu_e) + P(\nu_e \rightarrow \nu_\mu)} \\
&\approx \frac{\frac{8}{\sqrt{3}} \sqrt{\frac{m_e}{m_\mu}}}{\frac{16}{3} \frac{m_e}{m_\mu} + \left(\frac{\sin F_{\text{sun}}}{\sin F_{\text{atm}}} \right)^2} \sin F_{\text{sun}} .
\end{aligned} \tag{5.58}$$

Let us typically take the baseline length to be $L = 732$ km or $L = 7332$ km for a neutrino source at Fermilab pointing toward the Soudan mine in Minnesota or the Gran Sasso underground laboratory in Italy [102]. The mass-squared differences are chosen as (a) $\Delta m_{\text{sun}}^2 = 5 \times 10^{-5}$ eV² and (b) $\Delta m_{\text{sun}}^2 = 10^{-4}$ eV² versus the fixed $\Delta m_{\text{atm}}^2 = 10^{-3}$ eV². The behavior of \mathcal{A} changing with the beam energy E in the range $3 \text{ GeV} \leq E \leq 20 \text{ GeV}$ is shown in Fig. 5.2. We see that the asymmetry \mathcal{A} can be of $\mathcal{O}(0.1)$ to $\mathcal{O}(1)$. The matter effect on \mathcal{A} must be taken into account, in order to extract the genuine CP -odd parameters. For the model under consideration, the smallness of $|V_{e3}|$ (≈ 0.056) together with the maximal CP violating phase ($\phi \approx 90^\circ$) is expected to make the possible matter effect insignificant, and unable to completely fake the genuine CP -violating signals (see, e.g., Ref. [39] and references therein).

Note again that in the framework of three neutrino families the leptonic CP violation could be observable, if and only if the large-angle MSW oscillation is the true solution to the solar neutrino problem. The situation can dramatically change, when the four-neutrino mixing scheme is adopted to interpret solar, atmospheric and LSND neutrino oscillation data. The most general mixing matrix of four Majorana (3 active and 1 sterile) neutrinos consists of 6 rotation angles and 6 CP -violating phases, as already counted in section 3.1. However, the mixing angles of the sterile neutrino ν_s with the active neutrinos ν_μ and ν_τ are expected to be very small (see the arguments given in section 5.4). In this case one may approximate the 4×4 lepton flavor mixing matrix V , which links the neutrino mass eigenstates $(\nu_0, \nu_1, \nu_2, \nu_3)$ to the neutrino flavor eigenstates $(\nu_s, \nu_e, \nu_\mu, \nu_\tau)$, by taking

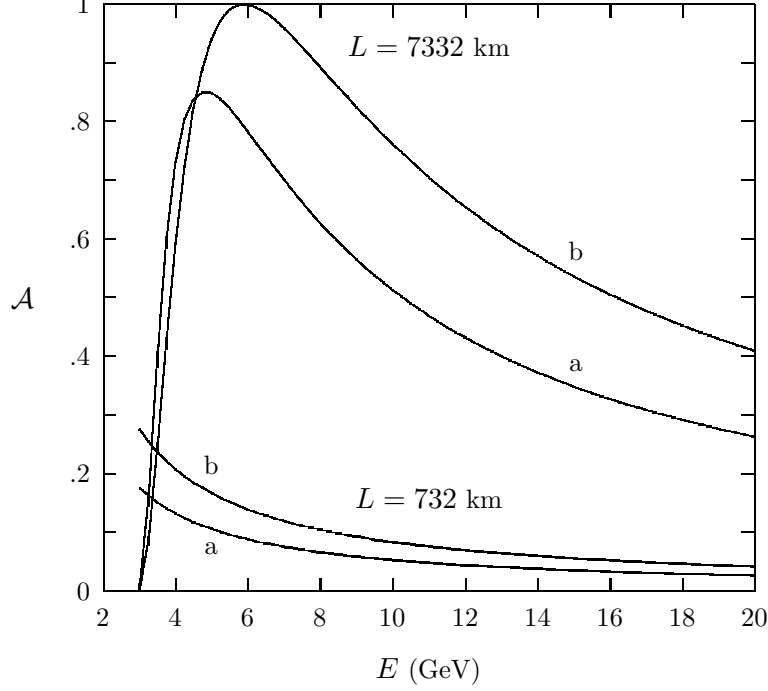


Figure 5.2: Illustrative plot for the CP -violating asymmetry \mathcal{A} between $\nu_\mu \rightarrow \nu_e$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ transitions in vacuum, changing with the neutrino beam energy E . Here (a) $\Delta m_{\text{sun}}^2 = 5 \times 10^{-5} \text{ eV}^2$ and (b) $\Delta m_{\text{sun}}^2 = 10^{-4} \text{ eV}^2$ versus the fixed $\Delta m_{\text{atm}}^2 = 10^{-3} \text{ eV}^2$ have typically been taken in the case of the baseline length $L = 732 \text{ km}$ or $L = 7332 \text{ km}$.

$\cos \theta_{02} \approx \cos \theta_{03} \approx 1$. Explicitly we have the following approximate form of V :

$$V \approx \begin{pmatrix} c_{01} & s_{01}^* & s_{02}^* & s_{03}^* \\ -s_{01} & c_{01} & s_{12}^* & s_{13}^* \\ V_{\mu 0} & V_{\mu 1} & c_{23} & s_{23}^* \\ V_{\tau 0} & V_{\tau 1} & -s_{23} & c_{23} \end{pmatrix}, \quad (5.59)$$

where $s_{ij} \equiv \sin \theta_{ij} e^{i\phi_{ij}}$ and $c_{ij} \equiv \cos \theta_{ij}$ (for $i, j = 0, 1, 2, 3$ and $i < j$), and the expressions of $V_{\mu 0}$, $V_{\mu 1}$, $V_{\tau 0}$ and $V_{\tau 1}$ can be obtained by using the leading order unitarity conditions. Based on this flavor mixing pattern as well as the mass spectrum given in (5.43), one may carry out an analysis of CP violation in $\nu_e \rightarrow \nu_\mu$, $\nu_\mu \rightarrow \nu_\tau$ and $\nu_\tau \rightarrow \nu_e$ oscillations. For example, the CP asymmetry between $\nu_\mu \rightarrow \nu_e$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ transitions reads

$$\begin{aligned} \Delta_{\mu e} &= P(\nu_\mu \rightarrow \nu_e) - P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) \\ &\approx 16 \mathcal{J}_l \sin F_{\text{atm}} \sin^2 F_{\text{LSND}}, \end{aligned} \quad (5.60)$$

where \mathcal{J}_l is the conventional rephasing-invariant measure of CP violation in the three-neutrino mixing framework; $F_{\text{LSND}} \propto \Delta m_{\text{LSND}}^2$ and $F_{\text{atm}} \propto \Delta m_{\text{atm}}^2$ describe the oscillation frequencies of LSND and atmospheric neutrinos, respectively. Note that the CP -violating effect arising from the interference between sterile and active neutrinos is negligible in

$\Delta_{\mu e}$, as a straightforward consequence of the approximation made in (5.59). Maximizing the contributions of ϕ_{ij} to the relevant CP -violating observables in an appropriate way, one can find that the signal of CP violation is much more significant and the matter effect is much smaller than those in the conventional three-neutrino mixing framework [101]. The most promising CP -violating asymmetry remains that between $\nu_\mu \rightarrow \nu_e$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ transitions, i.e., the asymmetry $\Delta_{\mu e}$ obtained in (5.60).

Let us end this section with one remark. The lepton flavor mixing angles depend in general not only on the ratios of charged lepton masses but also on those of neutrino masses, unless their contributions to flavor mixing are forbidden by some special flavor symmetries. Therefore the mixing factors of various neutrino oscillations are expected to have correlations with the corresponding neutrino mass-squared differences. In other words, $\sin^2 2\theta$ and Δm^2 may not be two completely independent parameters in the true theory of lepton masses and flavor mixing. All of the present model-independent analyses of CP violation in neutrino oscillations have tried to maximize the observable effects by adjusting the unknown parameters arbitrarily, i.e., regardless of the possible parameter correlation. We hope that progress in both theory and experiments would finally allow us to gain more insights into the problems under discussion and lead us to a full understanding of fermion mass generation and CP violation.

6 Concluding remarks

We have given an overview of phenomenological studies of fermion mass and flavor mixing schemes. Particular attention has been paid to the underlying flavor symmetries which can lead to realistic textures of quark and lepton mass matrices.

With the current experimental data on quark masses and flavor mixing angles, we find that only very few specific patterns of quark mass matrices are realistic. These patterns lead to different predictions for the sides and angles of the quark unitarity triangles, thus it is possible to distinguish one from another and to identify the “right” one by improved measurements of the quark mixing angles and of CP violation in the near future.

In comparison, the present experimental results for neutrino oscillations remain rather preliminary, although they do provide strong hints for possible patterns of lepton flavor mixing and the associated textures of lepton mass matrices. For illustration we have described several interesting schemes of lepton mass matrices, whose predictions can be examined in a variety of upcoming neutrino oscillation experiments. Nevertheless it should not be a surprise, if none of the currently proposed neutrino mass and mixing scenarios is realized by nature. But one may hope that at least some of the ideas explored in these attempts will survive the test of future measurements.

While the salient features of flavor mixing and CP violation have well been illustrated by a number of simple and instructive models of fermion mass matrices, some other interesting models proposed in the literature cannot be included in this article due to limitation of space. We therefore refer the reader to Refs. [31, 48, 103], in which the theoretical prospects for a deeper understanding of fermion mass generation and flavor mixing are described. We also refer the reader to Refs. [95, 104, 105], which elaborate the experimental prospects to determine the CP -violating phases of quark mixing, to pin

down the true mechanism of neutrino oscillations, and to measure the neutrino masses and lepton mixing parameters.

Certainly much more efforts are needed, both experimentally and theoretically, to gain a deeper insight into the generation of fermion masses, the pattern of flavor mixing, and the origin of CP violation in a unified picture of leptons and quarks. Let us finish with Max Jammer's remarks [106]:

“The modern physicist may rightfully be proud of his spectacular achievements in science and technology. However, he should always be aware that the foundations of his imposing edifice, the basic notions of his discipline, such as the concept of mass, are entangled with serious uncertainties and perplexing difficulties that have as yet not been resolved.”

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